I. Singer - Quadratic Functions in M-Theory & Topology 10/8/99
(w. M Hopkins) (& analysis & geometry)

Kaplansky rules for perfect colloquium: First five - landlady, next five - every undergrad, next five - every young mathematician, etc... next to last - experts strain, last five you strain

Within 5-brane effective action in M-theory hep-th 9710234

Σ Riemann surface, S = spin structures on Σ

- $H^1(\Sigma, \mathbb{Z}/2)$ - torsor.

Cup product $e_2 : H^1 \times H^1 \longrightarrow H^2 \cong \mathbb{Z}/2$

$\Gamma(L)$ sections.

<u>Riemann</u>: quadratic fn $e_*: S \to \mathbb{Z}/2$ $e(L) = \dim \Gamma(L) \bmod 2$ is stable under defs of Σ, & associated bilinear function is e_2.

$e_*(L \otimes \alpha \otimes \beta) + e_*(L \otimes \alpha) + e_*(L \otimes \beta) = \ldots$

Atiyah - explain via mod 2 index of Dirac operator $\bar\partial_s$ $s \in S$

$\bar\partial_s : L_s \to L_s \otimes \Lambda^{0,1}$ skew adjoint Fredholm operator.

- $e_*(L_s) = \dim \Gamma(L_s) \bmod 2$ is mod 2 index of $\bar\partial_s$.
- e_* quad function assoc to e_2.
- $s \in S$ cobordant to zero (spin cobordism) iff $e_*(s) = 0$ in $\Omega_2^{spin} \cong \mathbb{Z}_2$

Reformulate! $b: S \longrightarrow \Omega_2^{spin}$, $b(s) = \dim \ker \bar\partial_s \bmod 2$ $= e_*(s)$

$\to s \longmapsto$ quad fn. μ_s on $H^1(\Sigma, \mathbb{Z}_2)$

$\mu_s(\alpha) = b(s+\alpha) - b(s)$

(pick s trivialize $H^1(\Sigma, \mathbb{Z}_2)$ torsor $\leadsto$ quad fn on the latter).

Pontrjagin's mistake "$\pi_{n+2}(S^n) = 0$" ($= \mathbb{Z}/2$)

- can for map $S^{n+2} \to S^n$ get fibers Σ R.S, framed normal bundle
  for $S^1 \subset \Sigma$ element of π_1 want to surgery using framing ; assign ± 1 to S^1 if can or can't surgery
  — <u>not</u> linear, but <u>quadratic</u>.

$\rightsquigarrow$ Kervaire et al. study of quadratic fns. in topology.

$J_0 = I_c \quad \Sigma \rightsquigarrow M^r$ Riemannian of dim $4k+2$ $\quad J_0 = H^{2k+1}(\mathbb{R})/H^{2k+1}(\mathbb{Z})$

complex forms since $*^2 = -1$ in middle dimension

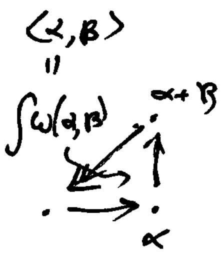
J_0 P.A.V. with translation inv. 2-form
$$\omega(\alpha,\beta) = \int_M \alpha \wedge \beta \qquad \alpha, \beta \in H^{2k+1}(\mathbb{R})$$

(Physics: dim 6, 10). $\quad$ - curvature of hermitian $\begin{array}{c}\mathcal{L}\\ \downarrow \\ J_0\end{array}$.

$\mathcal{L}$ determined by holonomy around geodesics representing $\pi_1(J_0) \simeq H^{2k+1}(M, \mathbb{Z})$
(know already holonomy around null-homotopic curves as integrals of curvature — i.e. know $\mathcal{L}$ mod flat bundles)

- geodesics are straight lines to lattice pts
$$\Rightarrow \varphi_{\mathcal{L}} : H^{2k+1}(M, \mathbb{Z}) \longrightarrow U(1)$$
which determines $\mathcal{L}$ with ω.

If $\mathcal{L}$ symmetric relative to $j \to -j$
$\Rightarrow \varphi_{\mathcal{L}}$ has values ± 1 $\longrightarrow$ quadratic function on $H^{2k+1}(M, \mathbb{Z}_2) \to \mathbb{Z}_2$



- associated form is cup product.

Symmetric $\mathcal{L} \longleftrightarrow$ quadratic functions
($k=0$: symmetric $\mathcal{L}$'s $\longleftrightarrow$ spinors)

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6 dimensions: (5 branes) realize these symmetric $\mathcal{L}$'s explicitly generalizing R.S. case ...

Physics: Q line bundle (Poincaré) on $\Sigma \times J_0$
$\Rightarrow$ elliptic family $\{\bar\partial_s \times I_q\}_{q \in Q}$

$\mathrm{Det}\,\bar\partial_s = \mathcal{L}_s$ has Quillen connection (nonzero hol section)
Holonomy around $S^1 \subset J_0$ is $e^{-2\pi i \xi}$
where ξ is η-invariant for Dirac operator
$\Leftrightarrow$ can find spin manifold X with spin boundary $\Sigma \times S^1$,
extend Q to X $(\tilde{Q})$ η = index of Dirac on X
$-\int_X \frac{c(\tilde{Q})^2}{2}$.

$\Rightarrow$ holonomy $2\pi i\, c(\tilde{Q}^2)/2$.

Thus Each $s \in S \longmapsto \mathcal{L}_s$ via det bundle with Quillen connection
- can write holonomy using cobordism.

Choice of symplectic basis $\rightsquigarrow$ quadratic fn via Θ fn...

Physics motivation: care about hol sections of this line bundle
(unique up to scale), whose $|\ |^2$ goes in partition fn.

Fix spin 6-manifold. Use Cheeger-Simons ($\leftrightarrow$ Beilinson-Deligne) cohomology.
Think of line bundles as holonomy along closed curves:
1-cycles $\mapsto U(1)$ which we know on boundaries.

Cheeger-Simons: $\hat{H}^{k-1}(M, S^1) = [(\chi, \omega)$ χ character of
smooth $k-1$ cycles, ω k-form s.t.
$\chi(\partial Z) = e^{2\pi i \int_Z \omega}$ $]$ Z k-chain

Define $d\chi$ to be ω.
e.g. • line bundles with $U(1)$ connections $\cong \hat{H}^1(M, S^1)$.

• $\hat{H}^0(M, S^1) = [(\varphi, \frac{\varphi^{-1} d\varphi}{2\pi i})$ $\varphi: M \to U(1)$ $]$
$\leftrightarrow$ "chiral boson"

• P $SU(N)$ bundle on M, A connection,
$C.S.(A) = \mathrm{tr}(A\,dA + \tfrac{2}{3} A^3)$.

$\Rightarrow (CS(A), c_2(A))$ determines element in $\hat{H}^3(M, S^1)$
- lift 3-cycle in M to P, $\chi(\sigma) = \int_{\tilde\sigma} C.S(A) \bmod \mathbb{Z}$

• WZW N_σ-model : $M = G$, $\check{H}^2(G) \xrightarrow[\simeq]{d} H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$

$\Sigma \to G$, $\partial X = \Sigma \ldots$

Exact sequences :

(*) $\quad 0 \to H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \to \check{H}^{k-1} \xrightarrow{d} \Omega_0^k \to 0$

closed k-forms with integral periods

• eg if G is closed 4th form with integral periods $\Rightarrow \exists$ character C "3-form" with $dC = G$

• If M has no torsion $H^{k-1}(M, \mathbb{R}/\mathbb{Z}) = J_0$ when $\dim M = 2k-2$, k even.

$\dim M = 6$, M spin $\Rightarrow$ $c_2(M)$ divisible by 2 ($Spin\,6 = SU(4)$)

$\hat{\lambda} \in \check{H}^3(M)$ 3-diff characters, $d\hat{\lambda} = c_2(M) \in \Omega_0^4(M)$

– Chern Simons form

So $\frac{c_2(M)}{2} \in \Omega_0^4(M)$, $\exists \frac{\hat{\lambda}}{2} \in \check{H}^3$.

two choices differ by $H^3(M, \mathbb{Z}/2)$.

<u>Theorem</u> Spin cobordism $\Omega_6^{\lambda/2} \simeq \mathbb{Z}/2$

(analogous to Ω_6^{Spin}, for $\simeq \mathbb{Z}/2$)

S_3 = such choices — $H^3(M, \mathbb{Z}/2)$-torsor

$\Rightarrow b_3 : S_3 \to \mathbb{Z}_2$ (cobordism class) a quadratic function.

$\to$ symmetric line bundle

Consequences : • change spin structure on M by $\alpha \in H^1(M, \mathbb{Z}/2)$

$\frac{\hat{\lambda}}{2} \mapsto \frac{\hat{\lambda}}{2} + \alpha^3$ – e.g. no change if no torsion !

• Hopkins : M^{4k+2} have map $\Omega^{\lambda/2}_{4k+2} \twoheadrightarrow \mathbb{Z}/2$

$\rightsquigarrow$ quadratic functions

$\mathcal{L}_{\lambda/2}$ = "a det line bundle" ?

$\varphi: \Sigma \to S^1$ $\quad \int \mathcal{D}\varphi \, e^{-\langle \varphi^{-1}d\varphi, \varphi^{-1}d\varphi \rangle}$

- Sum over H^1 classes + determinant terms for homotopically trivial part

Physics action $\quad \int \mathcal{D}B \, e^{-\langle dB, dB \rangle}$ $\quad$ over 2-gerbes

- Sum over $H^3(M, \mathbb{Z})$ + Sums over trivial type

Hitchin ~~geometry~~ of 3-forms

Pick $H \in \Lambda^3(M^6)$

For each $v \in V$ 6-dim vector space $H \in \Lambda^3 V^*$

look at $i(v) H \wedge H \in \Lambda^5(V^*)$

$\simeq V \otimes \Lambda^6(V^*)$

So get map $V \to V \otimes \Lambda^6(V^*)$

- now iterate $\to V \otimes (\Lambda^6(V^*))^2$

Assume volume element $\Rightarrow \Lambda^6 V^* \simeq \mathbb{C}$

$\Rightarrow$ get $T_H \in V \otimes V^*$ endomorphism.

Turns out $T_H = \alpha(H) \cdot I$ $\quad \alpha(H)$ number.

- $\boxed{\text{function on } \Lambda^3 V^*}$ which is 20-dim
- understand it under scale $\to$ 19-dim

Fix $\alpha(H) \Rightarrow$ 18-dim, two open orbits, $\simeq$

with ~~stabilizer~~ $SU(3)$ $\quad SO(3) \times SO(3)$

Connects with Seiberg-Witten on noncommutative geometry

[also work on Rozansky-Witten]