Quantum Groups - Y. Soibelman

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Observation - lattice versions of integrable systems are often integrable. Hope - conversely in limit lattice models give integrable models.

$\frac{2\text{ Russ}}{5} = 400l^\circ$ QISM : Baxter, Fadeev et al.

Hamiltonian approach - explain integrability from big commutative algebra.

Integrability - diagonalize (eigenvalues) H through big comm. algebra.

— "Big" "groups" of "symmetry", e.g. 2-dim CFT

- quantum group symmetry.

Let k be a field of char 0. A assoc alg / k.

Def. A is called Poisson if has $\{\}$ . . .

Poisson $\Rightarrow$ "symplectic" ~~...~~ $\pi \in \Gamma(M, \Lambda^2 TM)$

$\{f, g\} = \langle \pi, df \wedge dg \rangle$,

Schouten bracket on $\Lambda^\bullet(TM) = \oplus \Lambda^m$:

$[u_1 \wedge \dots \wedge u_m, v_1 \wedge \dots \wedge v_n] = \sum_{i,j} (-1)^{i+j} [u_i, v_j]\, u_1 \wedge \dots \hat{u}_i \dots \wedge v_1 \wedge \dots \hat{v}_j \wedge \dots v_n$

$[\Lambda^m, \Lambda^n] \subset \Lambda^{m+n-1}$, so shift $\Lambda^\bullet(TM)[1] = \alpha^\bullet$,

$[\alpha^m, \alpha^n] \subset \alpha^{m+n}$, graded Lie algebra.

$\pi \in \Lambda^2 \Rightarrow \pi \in \alpha^1$. Jacobi of $\{\}$ is equivalent to $[\pi, \pi] = 0$;

Define d.g. structure $d_\pi = [\pi, \cdot] : \alpha^\bullet \to \alpha^\bullet$,

$d_\pi^2 = 0$ since $[\pi, \pi] = 0$, so $(\alpha^\bullet, d_\pi)$ is a dg Lie algebra.

Prop If M is symplectic then this $(\alpha^\bullet, d_\pi)$ is (quasi) isom to $(\Omega^\bullet M, d)$

Thus to deform the symplectic manifold, we need only deform d.

$\Lambda^\bullet(TM)[1]$ with Schouten bracket, $d_\pi = [\pi, \cdot]$ (π Poisson structure) dg Lie alg.

Digression about quantization

Let $(M, \{\})$ be a Poisson manifold. A quantization of M is an assoc. algebra $A / \mathbb{C}[[h]]$ s.t. A is topologically free & complete in h-adic topology, $A/hA \simeq C^\infty(M)$ as Poisson algebras – isom. as assoc algebra, &
$\frac{ab-ba}{h} \bmod h = \{a \bmod h, b \bmod h\}$

Thus as $\mathbb{C}[[h]]$ modules $A \simeq C^\infty(M)[[h]]$ with multiplication $f * g = fg + \sum h^k c_k(f,g)$, with
$\{f,g\} = c_1(f,g) - c_1(g,f) = \{f,g\}$ for $f,g \in C^\infty(M)$..

Quantize the sheaf of smooth functions, this should be a local operation, i.e. $c_k(f,g)$ are bidifferential operators in f,g. $f * g|_U = f|_U * g|_U$.

(M,ω) symplectic, then Th. (Lecomte – deWilde, 1983).
1. M admits a quantization 2. All the quantizations (mod equivalence) are parametrized by formal paths $H^2(M)[[h]]$...
more precisely, given a quantization get its canonical class $cl(A) = [\omega] + \sum h^k \alpha_k$, $\alpha_k \in H^2(M)$.
In particular $(M, \omega) = (\mathbb{R}^{2n}, \sum dp_i \wedge dq_i)$ then its quantization is unique:
$\pi_0 = \sum \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$. $e^{\frac{h}{2}\pi_0}$ is an infinite series with coefficients polyvector fields, act

$$\begin{array}{c} e^{\frac{h}{2}\pi_0}(f \otimes g) \in (C^\infty(M) \otimes C^\infty(M))[[h]] \\ \downarrow \text{mult} \\ f * g \longrightarrow C^\infty(M)[[h]] \end{array}$$

– Moyal quantization.

Darboux – any (M,ω) is locally $\simeq \mathbb{R}^{2n}, \sum dp_i \wedge dq_i$.
So any symplectic manifold admits a local quantization.. how to glue together? Lecomte-deWilde 1990 – obstruction, in $H^3(M)$, vanishes. Refinement by Deligne – nonab. cohomology, consider $U \to \mathcal{C}(U)$ category of all quantizations of $C^\infty(U)$, which is a groupoid (any two are isomorphic). $\rightsquigarrow$ sheaf of groupoids, or a gerbe...

Fedosov quantization (1985, 1991) – nonlocal approach.

— sheaf of noncommutative algebras on a symplectic manifold. Locally sheaf of differential operators — use polarization to split into p and q parts. — double the # of variables, get sheaf not gerbe.. Weyl algebra double: $W =$ $p_1, \dots p_n, q_1 \dots q_n, x_1 \dots x_n, y_1 \dots y_n$, $[x_i, y_j] = \delta_{ij} h$. Dependence on q, p. Get flat connection on this sheaf, quantization is then flat sections of ∇_{Fed} on W. ■ Digression.

M Poisson — get canonical local models only where rank of $\overset{\vee}{\pi}_x : T_x^* M \to T_x M$ is constant, factoring Symp × Null foliation.

<u>Def</u> $f \in C^\infty(M) \longrightarrow H_f = \{f, \cdot\}$ Hamiltonian vector field.

A submanifold $N \subset M$ is called a Poisson submanifold if $H_f(n)$ is tangent to N, $n \in N$ for all $f \in C^\infty(M)$.

A Hamiltonian curve is the integral curve of a Hamiltonian vector field. We say $x \sim y$ $(x, y \in M)$ if $\exists$ a piecewise Hamiltonian curve connecting x, y.

<u>Def</u> An equivalence class of a point is called a symplectic <u>leaf</u>.

<u>Prop</u> Let $S \subset M$ be a symp. leaf. Then i) S is a Poisson submanifold ii) S is symplectic. iii) M is a union of its symplectic leaves

<u>Pf</u> (Sketch) Compute Lie derivative of $\overset{\vee}{\pi}_x : T_x^* M \to T_x M$, show its rank is constant along S. ...

G has a (noncanonical) Poisson structure related to quantum groups & s.t. $G/B = \bigcup$ Schuberts is symplectic leaf decomposition.

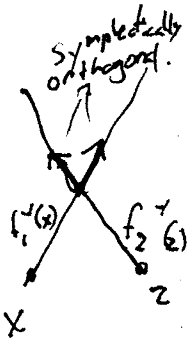
Def A Dual pair is a diagram

$$(*) \quad \begin{tikzcd} & S \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

where 1). S is symplectic, P_1, P_2 Poisson
2) f_i are morphisms of Poisson manifolds.
3). $f_1^*(C^\infty(P_1))$ and $f_2^*(C^\infty(P_2))$ centralize each other in $C^\infty(S)$ in Poisson sense.

Def A dual pair is called full if f_i are submersions.

Theorem Let $(*)$ be a full dual pair. Then for any $x \in P_1$ the blow up $M_x = f_2 f_1^{-1}(x)$ is a symplectic leaf in P_2.



Pf First M_x is a manifold. Now consider $f_1^{-1}(x) \ni y$.
$(\varphi_1 \ldots \varphi_n)$ local coords at x $\Rightarrow f_1^*(\varphi_i) \in C^\infty(x)$.
$T_y(f^{-1}(x)) = \{ v \in T_y S \mid \langle d f_1^*(\varphi_i), v \rangle = 0 \ \forall i \}$
On the other hand take $(\psi_1 \ldots \psi_{n_2})$ local coords at $z = f_2(y)$.
$f_2^*(\psi_i) \in C^\infty(S)$. Now $\langle df_1^*(\varphi_i), H_{f_2^*(\psi_j)} \rangle = 0$
so these functions ($\Leftrightarrow$ Hamiltonian fields) sit in $T_y f_1^{-1}(x)$.
Given $v \Rightarrow g \in C^\infty(S)$ s.t. $v = H_g$. So if $v \in T_y f_1^{-1}(x)$ then g is in the centralizer of $f_1^*(C^\infty(P_1))$ so g comes from P_2. ∎

Lemma W a symplectic vect space, $V \subset W$ a subspace, then $V / V \cap V^\perp$ carries natural symplectic structure.

Take $S = T^*G$, have two actions of G from left, right

$$\begin{tikzcd} & S=T^*G \arrow[dl, "\text{left}"'] \arrow[dr, "\text{right}"] & \\ \mathfrak{g}^* & & \mathfrak{g}^* \end{tikzcd}$$

. $\underline{f_2 f_1^{-1}(x) = \text{coadjoint orbit}}$!

Definition Let G be a Lie group which is also a Poisson manifold. G is said to be a Poisson-Lie group if mult: $G \times G \to G$ is Poisson.
$\Rightarrow \mathfrak{g}^*$ Lie algebra.

Def A pair $(\mathfrak{g}, \varphi)$ is called a Lie bialgebra if a. $\mathfrak{g}$ is a Lie algebra b. $\varphi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ linear map s.t. $\varphi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket. c. φ is 1-cocycle wrt $\cdot$ ad$_\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$: $\varphi([x,y]) = \cancel{x} \cdot \varphi(y) - y \cdot \varphi(x)$.
$x \cdot a \otimes b = [x, a] \otimes b + a \otimes [x, b] = \mathrm{ad}\, x (a \otimes b)$

Ex. $\mathfrak{g} = sl_2 \mathbb{C}$. $\varphi(H) = 0$, $\varphi(X^\pm) = X^\pm \wedge H$.
Exercise: this is a Lie bialgebra.
$\mathfrak{g}$ complex semisimple. $D = (d_1 \ldots d_n)$ $d_i a_{ij} = d_j a_{ji}$
$\varphi(H_i) = 0$ $\varphi(X_i^\pm) = d_i X_i^\pm \wedge H_i$.

Last time we did double construction of Lie bialgebra $\rightsquigarrow D(\mathfrak{g})$ Lie bialg.

$\Leftrightarrow$ Manin triple $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ $\to$ Lagrangian, $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$.

$\langle, \rangle_{\text{nondegen}}$

$\Rightarrow$ Lie bialgebra structures on $\mathfrak{p}_\pm$ ..

Poisson groups] Manin pairs: $(\mathfrak{p}, \mathfrak{p}_+)$ .. $\exists\, \mathfrak{p}_-$?

Ex. start with curve C with fin subset S with local params $x_1 \dots x_n$. $\mathfrak{g}$ fin dim simple, $\langle, \rangle$ - Killing $\Rightarrow$ Manin ~~pair~~

$= \mathfrak{p} = \mathfrak{g}_S(A)$ adeles on C, singularities at S

$\mathfrak{p}_+ =$ rational adeles, singularities at S

$\hookrightarrow$ come from rational functions on C,

$\mathfrak{p}_+$ Lie subalgebra

Invariant forms: $\langle (a_x), (b_x) \rangle = \sum_{x \in C} \operatorname{Res}_x \langle a_x, b_x \rangle_{\mathfrak{g}}$

$\mathfrak{p}_+$ isotropic, sum of residues $= 0$

Can this be made a Manin triple ?

Ans.: not for genus >1, genus $=1$ $\mathfrak{g} = \mathfrak{sl}_n$ only can be done. $g = 0$ always can.

Poisson groups G fin dim Lie group, Poisson, $\mu: G \times G \to G$ Poisson. G alg group: $k[G]$ Hopf algebra

Def Let A be a comm. Hopf algebra which is also Poisson. A is Poisson-Hopf if comult. $\Delta: A \to A \otimes A$ is Poisson

Multiplicativity Prop Let G be a P-Lie group $\pi \in \Lambda^2(G, TG)$ corresp bivector

Then $\pi(g_1 g_2) = l_{g_1 *} \pi(g_2) + r_{g_2 *} \pi(g_1)$

Pf $\Delta: C^\infty(G) \to C^\infty(G \times G)$ Poisson $\Longrightarrow$

$\Delta(\{\varphi, \psi\})(g_1, g_2) = \{\Delta\varphi, \Delta\psi\}_{G \times G} = \{\varphi(g_1 g_2), \psi(g_1 g_2)\}$,

$\{\varphi, \psi\} = \langle \pi, d\varphi \wedge d\psi \rangle$.

Cor G Poisson-Lie, then e is a symplectic leaf ($\pi(e) = 0$).

Set $\eta: G \to \Lambda^2 \mathfrak{g}$. $\eta(g) = (r_{g^{-1}})_* \pi(g)$ is 1-cocycle

$\eta(g_1 g_2) = \eta(g_1) + \operatorname{Ad}_{g_1} \eta(g_2)$.

$\mathfrak{g}$ becomes Lie bialgebra, $d\eta = \varphi: \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ defines bialg. structure $\{\varphi, \psi\}(g) = \sum \eta^{\mu\nu}(g)\, \partial_\mu \varphi\, \partial_\nu \psi$.

Can always write Lie bialgebra from quadratic equations corresp to Jacobi — comes from dgla, generated by double $\mathfrak{g} \oplus \mathfrak{g}^*$, get standard complex

Jacobi $\Leftrightarrow d^2 = 0$

$\mathfrak{g}/k$ fin-dim Lie alg. $k[\mathfrak{g}^*]$ has Kirillov-Kostant Poisson ($\mathfrak{g}^*$ commutative Lie algebra) $\Rightarrow$ $k[\mathfrak{g}^*]$ Poisson-Hopf.
$\Rightarrow$ $\mathfrak{g}^*$ is an algebraic Poisson group
— its quantization is $U(\mathfrak{g})$

$G = SL_2\mathbb{C}$, $\mathbb{C}[G] = \{t_{ij},\ t_{11}t_{22} - t_{12}t_{21} = 1\}$
Poisson bracket $\{t_{11}, t_{12}\} = -t_{11}t_{12}$, $\{t_{11}, t_{21}\} = -t_{11}t_{21}$,
$\{t_{12}, t_{22}\} = -t_{12}t_{22}$ $\{t_{21}, t_{22}\} = -t_{21}t_{22}$, $\{t_{11}, t_{22}\} = 2t_{12}t_{21}$,
$\{t_{12}, t_{21}\} = 0$ $\Longleftrightarrow$ Standard Poisson-Lie structure
Consider now functions (rational) with sings at divisor $t_{12}t_{21} = 0$ — $\frac{f(g)}{t_{12}^m t_{21}^n}$
$\Rightarrow$ (Poisson) central element $\frac{t_{12}}{t_{21}}$ $\Longrightarrow$ $\frac{t_{12}}{t_{21}} = \text{const}$ are symplectic leaves.

Duality: G Poisson-Lie

$$\begin{array}{ccc} G & \longrightarrow & G^* \\ \uparrow & & \uparrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}^* \end{array}$$

$SU_2 \subset SL_2\mathbb{C}$
$SU(2)^*$: $\mathcal{D}(SU(2)) = sl_2 = k \oplus a \oplus n$
$\Rightarrow AN$

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1-1 corresp between connected simply con. P-Lie groups & Lie bialgebras
$\Rightarrow$ construct double $\mathcal{D}(G)$, which is P.Lie
Locally looks like $G \times G^*$.
$\{I_\alpha\}$ basis of $\mathfrak{g}$, $\{I^\alpha\}$ dual basis of $\mathfrak{g}^*$

$$\{\varphi, \psi\}_{\mathcal{D}(G)} = \frac{1}{2}\sum_\alpha \{(\partial_\alpha\varphi\,\partial^\alpha\psi - \partial^\alpha\varphi\,\partial_\alpha\psi) - (\partial'_\alpha\varphi\,\partial'^\alpha\psi - \partial'^\alpha\varphi\,\partial'_\alpha\psi)\}$$

I_α $\to \partial_\alpha$ right invariant v.f.
$\to \partial'_\alpha$ left invariant v.f. etc.

$\Rightarrow$ P-Lie structure on $\mathcal{D}(G)$ corresp to Lie bialgebra $\mathcal{D}(\mathfrak{g})$.

Consider the same bracket, but with $+$ between the two commutators
$\Rightarrow$ another Poisson structure on $\mathcal{D}(G)$, it is symplectic in the neighborhood of $1 \in \mathcal{D}(G)$.
This Poisson manifold is denoted by $\mathcal{D}(G)_s$

Theorem

$$\begin{array}{ccc} & \mathcal{D}(G)_s & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ G^*\backslash\mathcal{D}(G)_s & & \mathcal{D}(G)_s/G^* \end{array}$$

1) There are natural Poisson structures on all objects

If we identify locally ${}_{G^*}\backslash^{\mathcal{D}(G)_s}$, $\mathcal{D}(G)_s/_{G^*}$ with G then these Poisson structures coincide with the P-Lie structure on G

2). This is a full dual pair (locally near 1)

Locally $\mathcal{D}(G) = G\cdot G^* = G^*\cdot G$. $hg = g' h'$ – left action $G^* \times G \to G$, $g \mapsto g'$, & viceversa right action G on G^*
– dressing actions.

Prop Symplectic leaves in G are (locally) invariant wrt dressing action of G^*. If the dressing action is defined globally then symplectic leaves = dressing orbits.
– follows from dual pair description of symplectic leaves as blowups of points

Examples G, $\{\} = 0$ P-Lie. $\mathfrak{g}^* = G^*$. $\{\}_{\mathfrak{g}^*}$ = K-K structure
$\mathcal{D}G = T^*G = G \ltimes \mathfrak{g}^*$ (coadjoint) – not symplectic.
$\mathcal{D}G_s = T^*G$ symplectically, $\mathfrak{g} \leftarrow T^*G \to \mathfrak{g}^*$
Dressing: $g \cdot l = Ad_g^*(l)\cdot g$,
symplectic leaves are coadjoint orbits.

Ex K simple compact $\subset G$ complex simple. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
$SL_n = SU_n \cdot (AN)$ A: diagonal entries $a_i > 0$.
$G = \mathcal{D}(K) = K\cdot K^* = KAN$
Dressing $AN \times K \to K$ defined globally.
Symplectic leaves are projections of double classes $G^* g G^* \cap G$
$AN \backslash K / AN \cap K$ symplectic leaves – related to Bruhat.

Dressing vector fields $\alpha \in \mathfrak{g}^* \to \alpha_l, \alpha_R$ left/right inv. forms on $\mathfrak{g}^*$.

Dressing transformation: action $G^* \times G \to G$ (left), also right.

$\xi \in \mathfrak{g}^* \rightsquigarrow (L_g)_* \xi \in \Omega^1(G)$ left invariant one-form
$g \in G$

Using Poisson isomorphism $\check{\pi} : T^*G \to TG \Rightarrow$

$\rho_l(\xi)$ left dressing vector field.. if can be integrated get group action.. always true if G compact.

Theorem orbits of the dressing action are symplectic leaves.

e.g. $SU(2)$: two sorts of leaves

a. $T \ni t$ diagonal matrix – zero dim leaves.

b. $t \in T$. $S_t = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2) \,\middle|\, \arg b = \arg t \right\}$

two dimensional leaves.. closure of each contains whole circle of 0-dim leaves.

e.g. $\mathcal{D}(G) = G \times G$, G acts on itself with symplectic leaves conjugacy classes.

$(\mathfrak{g}, \varphi)$ Lie bialgebra. $\varphi \in H^1(\mathfrak{g}, \Lambda^2 \mathfrak{g})$ can be coboundary $\varphi = \partial r$, $r \in \Lambda^2 \mathfrak{g}$

$\varphi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ Lie bracket gives condition for r:

$\langle r, r \rangle \equiv [r^{12}, r^{13}] + [r^{13}, r^{23}] + [r^{13}, r^{23}] \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$

$\mathfrak{g}$-invariant

$r^{ij} \in U\mathfrak{g} \otimes U\mathfrak{g} \otimes U\mathfrak{g}$

- mCYBE.

$\langle r, r \rangle = 0$: CYBE. r – classical r-matrix

Prop $(\mathfrak{g}, \varphi)$, $\varphi = \partial r$ defines Lie bialgebra iff mCYBE.

pf. $\alpha, \beta, \gamma \in \mathfrak{g}^*$, $x \in \mathfrak{g}$ then

$\langle \mathrm{Jacobi}(\alpha, \beta, \gamma), x \rangle = \langle \mathrm{ad}_x(\langle r, r \rangle), \alpha \otimes \beta \otimes \gamma \rangle$ ∎

$r \Rightarrow (\mathfrak{g}, \partial r) \to G$ Poisson-Lie group

Take I_μ basis of $\mathfrak{g} \Rightarrow$ right inv ∂_μ, left inv ∂'_μ.

$r = \sum r^{\mu\nu} I_\mu \otimes I_\nu$

$\Rightarrow \{\varphi, \psi\}_G = \sum_{\mu,\nu} r^{\mu\nu} (\partial_\mu \varphi \, \partial_\nu \psi - \partial'_\mu \varphi \, \partial'_\nu \psi)$

Suppose $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies CYBE (not nec. skew-symmetric)

Does $\varphi = \partial r$ still give Lie bialgebra?

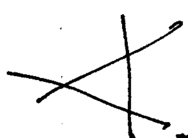
$\varphi(x) = [x \otimes 1 + 1 \otimes x, r]$

Assume $u = r^{12} + r^{21}$ is $\mathfrak{g}$-invariant ($r^{12} = r$, r^{21} permuted).
$\tilde{r} = r - \frac{1}{2}u$ is skew symmetric, $\in \Lambda^2 \mathfrak{g}$
$\langle \tilde{r}, \tilde{r} \rangle = \frac{1}{4}[u^{12}, u^{23}] \in \Lambda^3 \mathfrak{g}$ — needs to be <u>invariant</u>.

Can we reconstruct r satisfying CYBE (possibly not skew) from $\tilde{r}$ satisfying mCYBE? Yes for $\mathfrak{g}$ simple! $(\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$ is 1-dim.

Terminology: 1. $(\mathfrak{g}, \partial r)$ is a <u>coboundary</u> Lie bialgebra
2. Triangular Lie bialgebra: $\langle r, r \rangle = 0$.
3. Quasitriangular: $\langle r, r \rangle = 0$, $r \in \mathfrak{g} \otimes \mathfrak{g}$
not nec. skew-symmetric.

QYBE also called triangle equation: scattering condition for 3 moving lines.



<u>Example</u> 1. $\mathfrak{g}$ Lie bialgebra, $\mathcal{D}(\mathfrak{g})$ is a q.t. Lie bialgebra
$r = \sum I_\alpha \otimes I^\alpha$ $\quad I^\alpha$ dual basis to I_α.
$\Rightarrow r^{12} + r^{21}$ invariant ...

2. $\mathfrak{g} = sl(2)$. $\varphi(H) = 0$, $\varphi(X^\pm) = X^\pm \wedge H$
sl_2 is almost a double of Borel b_+:
$\mathcal{D}(b_+) \simeq \mathfrak{g} \times \mathfrak{h} \twoheadrightarrow \mathfrak{g}$ gives q.t. structure on $\mathfrak{g}$. $\quad r = X^+ \otimes X^- + \frac{1}{4} H \otimes H$.
$r^{12} + r^{21}$ invariant.
(True for any K-M). $\quad \tilde{r} = \dfrac{X^+ \otimes X^- - X^- \otimes X^+}{2}$
$r + \frac{1}{2}$ Casimir...
$\mathfrak{g}$ – any simple $/\mathbb{C}$, standard $\varphi(X_i^\pm) = X_i^\pm \wedge H_i$,
$\varphi(H_i) = 0$ is again q.t.
$\{X_\alpha\}_{\alpha \in \Delta}$ Cartan basis, $(\,,\,)$ invariant
$(X_\alpha, X_{-\alpha}) = 1$.
$\tilde{r} = \sum_{\alpha > 0} \dfrac{X_\alpha \otimes X_{-\alpha} - X_{-\alpha} \otimes X_\alpha}{2}$ $\qquad r = \sum X_\alpha \otimes X_{-\alpha} + \sum H_\mu \otimes H_\mu$
in orthonormal basis.

<u>Exercise</u> Classify all 2-dim Lie bialgebras.

3. on sl_2 $\quad r = \frac{1}{2}(H \otimes X^+ - X^+ \otimes H)$ another Lie bialgebra structure, in fact triangular.

5 Choices: $\mathfrak{g}$ = comm or $[h, x] = x$, same for $\mathfrak{g}^*$, on comm have 0 structure and Lie-Kirillov one.

<u>Belavin-Drinfeld thm</u> $\mathfrak{g}/\mathbb{C}$ simple, $(\mathfrak{g},\varphi) \Rightarrow \varphi = \partial r$ ($H' = 0$)

1. $r^{12}+r^{21}=0$, $\langle r,r\rangle=0$ (triangular)
2. $r^{12}+r^{21}\neq 0$, $\langle r,r\rangle=0$

B-D classifies class 2:

<u>Discrete parameters</u> triples $(\Gamma_1, \Gamma_2, \tau)$ s.t.

$\Gamma_i \subset \Pi$ subset of simple roots. $\tau : \Gamma_1 \to \Gamma_2$ bijection.

s.t. a. $(\tau(\alpha), \tau(\beta)) = (\alpha, \beta)$: isometry

b. for any $\alpha \in \Gamma_1$ $\exists\, k \geq 1$ integer s.t. $\tau^k(\alpha) \notin \Gamma_1$

$\Rightarrow$ "admissible triple"

<u>Continuous parameters</u>

$r_0 \in \mathfrak{h}\otimes\mathfrak{h}$ satisfying a. $r_0^{12}+r_0^{21} = t_0$ (canonical element)

$t_0 \Longleftrightarrow$ scalar product $(,)$

b. $(\tau(\alpha)\otimes id)(r_0) + (id \otimes \alpha)(r_0) = 0$ $\alpha \in \Gamma_1$.

$\mathbb{Z}\Gamma_i$ - lattice spanned by Γ_i. $\hat{\Gamma}_i = \mathbb{Z}\Gamma_i \cap \Delta$

$\alpha, \beta \in \Delta$ set $\alpha < \beta$ if $\beta = \tau^k(\alpha)$ $k \geq 1$

<u>Theorem</u> (1) Fix discrete & cont. data. Then the following r satisfies $\langle r,r\rangle=0$, $r^{12}+r^{21}\neq 0$.

$$r = r_0 + \sum_{\alpha>0} X_\alpha \otimes X_{-\alpha} + \sum_{\substack{\alpha,\beta>0\\ \beta>\alpha}} (X_\alpha \otimes X_{-\beta} - X_{-\beta}\otimes X_\alpha)$$

(2) Any solution $\langle r,r\rangle=0$, $r^{12}+r^{21}\neq 0$ can be transformed into this form by an automorphism of $\mathfrak{g}$ (& scaling).

Belavin-Drinfeld theorem - describes CYBE solutions $\langle r,r\rangle=0$, $r^{12}+r^{21}\neq 0$
$r\in\mathfrak{g}\otimes\mathfrak{g}$, $\mathfrak{g}$ simple / $\mathbb{C}$.

$$r=r_0+\sum_{\alpha>0}X_\alpha\otimes X_{-\alpha}+\sum_{\substack{\alpha,\beta>0\\ \beta<\alpha}}(X_\alpha\otimes X_{-\beta}-X_{-\beta}\otimes X_\alpha)$$

$r_0\in\mathfrak{h}\otimes\mathfrak{h}$ $\alpha\in\hat{\Gamma}_1$, $\beta\in\hat{\Gamma}_2$ subsets of the Dynkin diagram

How about skewsymmetric CYBE solutions ?

- much more complicated. Assume $\mathfrak{p}\subset\mathfrak{g}$ subalgebra, $r\in\mathfrak{p}\wedge\mathfrak{p}$ nondegenerate (as operator $\mathfrak{p}\to\mathfrak{p}^*$)

<u>Prop</u> r defines a 2-cocycle on $\mathfrak{p}$ (from CYBE).

- $T\in V\otimes V$, $\langle\,\rangle$ on $V\Rightarrow B(x,y)=\langle T,x\otimes y\rangle$

$\Rightarrow$ $B=B_r$, CYBE for $r\Leftrightarrow B$ satisfies cocycle condition
$B([X,Y],Z)+$ cyclic permutations $=0$.

B 2cocycle is a coboundary if $\exists\, l\in\mathfrak{p}^*$ s.t. $B(x,y)=l([x,y])$

Problem: $\Rightarrow$ Find $\mathfrak{p}\subset\mathfrak{g}$ and $l\in\mathfrak{p}^*$ s.t. $B_l(x,y):=l([x,y])$ is non-degenerate..

<u>Def</u> we call the Lie alg $\mathfrak{p}$ Frobenius if there exist such l, B_l nondegen

<u>Ex.</u> $\mathfrak{g}=sl_2$, $\mathfrak{p}=\mathfrak{b}_+=\langle X^+,H\rangle$. $B(aX^++bH,cX^++dH)$
$=\det\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $\rightsquigarrow r=X^+\otimes H-H\otimes X^+$

$l(aX^++bH)=\frac{1}{2}a$ ...

Usually all of $\mathfrak{g}$ isn't Frobenius, but has Frob subalgebras

<u>Prop</u> If $\mathfrak{g}$ is semisimple / k char $0\Rightarrow\mathfrak{g}$ is not Frobenius.

There are Frobenius algebras in category of Lie algebras: $\langle[a,b],c\rangle=\langle a,[b,c]\rangle$.
- algebras over Lie operad.

<u>Proof</u> $l\in\mathfrak{g}^*$ invariant $\rightsquigarrow B([X,Y],Z)=-B(Y,[X,Z])$
$\rightarrow$ 2 terms from Jacobi $l([X,Y],Z]+l([Y,[X,Z]])=0$
$=l([Z,[X,Y]])=0$ but anything can be written this way if $\mathfrak{g}$ semisimple...

- Wild problem, complicated moduli, contains as subproblem classification of Frobenius Lie algebras.

Solution of CYBE $\Rightarrow$ $(\mathfrak{g}, \varphi = \partial r)$ Lie bialgebra

Compact Lie bialgebras : get complete classification. sum of semisimple part & commutative part. Commutative subalgebra can't be Frobenius $\ell([X,Y]) \equiv 0$ — so can assume k is semisimple. $p \subset k \Rightarrow p$ is compact, either commutative or semisimple, p can't be Frobenius $\Rightarrow$ no skew-symmetric solution of CYBE!

So Belavin-Drinfeld classification applies :

our data : $\mathfrak{g}$ complex simple, $k \subset \mathfrak{g}$ maximal compact. k = stable points of $\omega_0 : \mathfrak{g} \to \mathfrak{g}$, $X_\alpha \mapsto -X_{-\alpha}$, $\omega_0|_H = -\mathrm{Id}$

— look for ω_0 - invariant solutions of B-D r's.

$\Rightarrow \Gamma_1 = \Gamma_2 = \emptyset$

$\Rightarrow r = r_0 + \sum_{\alpha > 0} X_\alpha \otimes X_{-\alpha}$, with $r_0^{12} + r_0^{21} = t_0$. t_0 is Cartan part of Casimir.

— unique r_0 up to adding $u \in \Lambda^2 \mathfrak{h}$

Now solution $\langle r, r \rangle = 0, r^{12} + r^{21} \neq 0 \iff$ consider $\tilde{r} = r - \frac{1}{2} V$, $V = r^{21} - r^{12}$, $\tilde{r}$ skew-symmetric, satisfies mCYBE.

Theorem Let $a \in \mathbb{R}$, $u \in \Lambda^2 \mathfrak{h}$. Then
$\tilde{r}(a, u) = \frac{a i}{2} \sum_{\alpha > 0} X_\alpha \wedge X_{-\alpha} + u$ is solution of mCYBE, & any solution of mCYBE for k is of this form.

Proof To solve $r_0^{12} + r_0^{21} = t_0$, take basis $I_\mu \subset \mathfrak{h}$,

$r_0 = \sum I_\mu \otimes I_\mu$, $r = \sum I_\mu \otimes I_\mu + \sum X_\alpha \otimes X_{-\alpha}$

$r^{12} + r^{21} = 2 \sum I_\mu \otimes I_\mu + \sum_{\alpha > 0} X_\alpha \otimes X_{-\alpha} + X_{-\alpha} \otimes X_\alpha$

$\tilde{r} = \sum X_\alpha \wedge X_{-\alpha}$, put i in front to make ω_0-invariant

Can replace $r_0 \to r_0 + u$ ∎

Standard Lie bialgebra on $\mathfrak{g}$: $\varphi(X_i^{\pm}) = X_i^{\pm} \wedge H_i$, $\varphi(H_i) = 0$
— comes from $a = 1$, $u = 0$ get the standard structure

$K = \mathrm{Lie}^{-1} k \Rightarrow$ Poisson Lie group $K(a,u)$.

What are the symplectic leaves : $u=0$ case (up to scalar)

$\sim$ $K(1,0) = K$ standard case

$SU(2)$ - symplectic leaves were 1. $t \in T$ pts

2. $S_t = \{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ $t \in T$, arg b fixed = arg t $\}$

discs e.g. $S_{-1} = \{ \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \mid b < 0 \}$

Take i = vertex of the Dynkin diagram of $\mathfrak{g} \Rightarrow \varphi_i : SU(2) \to K$

$(sl_2)_i \hookrightarrow \mathfrak{g}$. $\Sigma_i = \varphi_i (S_{-1})$

$\Rightarrow \Sigma_i$ is a symplectic leaf in K since

this is morphism of Poisson-Lie groups.

Generally take $w \in W$, $w = s_{i_1} \dots s_{i_k}$ reduced expression,

$\Sigma_w := \Sigma_{i_1} \cdot \Sigma_{i_2} \cdots \Sigma_{i_k}$ product set.

Theorem.. 1) Σ_w is a symplectic leaf in K

2) $t \in T$ maximal torus is a zero-dim symplectic leaf 3) Any symplectic leaf $S \subset K$ can be represented as $S = \Sigma_w \cdot t$ for some w, t.

$(w,t) \longleftrightarrow N(T)$, parametrizes symplectic leaves

K/T flag manifold $\Rightarrow$ Bruhat decomposition, Bruhat cells are images of Σ_w.

A_0 commutative Poisson algebra / k

Quantization - a topological assoc algebra $A/k[[h]]$

s.t. $A/hA \simeq A_0$, $\frac{ab-ba}{h} \bmod h = \{a \bmod h, b \bmod h\}$

& A is free as a top. $k[[h]]$ - module.

$A_0 = C^\infty(M)$ (M,ω) symplectic $\Rightarrow$ $\exists$ quantization

Poisson case - unknown in general, even locally

G Poisson-Lie group - never symplectic

Ex. $\mathfrak{g}^*$ coalgebra of a Lie alg. $\mathfrak{g}$: $\mathfrak{g}^* = G^*$ dual Lie group with trivial Poisson structure

$\mathfrak{g}^*$ has linear Poisson structure. $k[\mathfrak{g}^*]$ Hopf algebra

- require the quantization A to be a (compatible) Hopf algebra as well $\Rightarrow$ $U\mathfrak{g}[[h]] \supset A$ = generated by elements $x \in \mathfrak{g}$ with relation $xy - yx = h[x,y]$

- A topological Hopf algebra / $k[[h]]$ .. quantization

More generally: <u>Idea</u>: consider the dual Hopf algebra,

$U\mathfrak{g}$ - dual to functions $Fun(G)$

$C^\infty(G) \times U\mathfrak{g} \to \mathbb{C}$ as diff op pairing evaluated at e

- dual to functions on the formal group

If $(\mathfrak{g}, \varphi)$ is a Lie bialgebra then $U\mathfrak{g}$ carries a co-Poisson Hopf alg. structure

$\delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ s.t. $\delta|_\mathfrak{g} = \varphi$

$\delta(ab) = \delta(a)\Delta_0(b) + \Delta_0(a)\delta(b)$

where $\Delta_0 : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ is standard comultiplication

<u>Def</u> A quantization of a Lie bialg $(\mathfrak{g}, \varphi)$ is a topological Hopf alg $(A, \Delta)/k[[h]]$ s.t. $A/hA \simeq U\mathfrak{g}$ as a Hopf algebra, $\frac{\Delta(a) - \Delta'(a)}{h} \bmod h = \delta(a \bmod h)$

$(\Delta : A \to A \hat{\otimes} A$

where $\Delta' = \sigma \circ \Delta$, $\sigma(a \otimes b) = b \otimes a$ reverse comult.

& A is topologically free $k[[h]]$ module

<u>Question</u> Is there a quantization of an arbitrary Lie bialgebra $(\mathfrak{g}, \varphi)$?

(Raised by Drinfeld)

Theorem (Etingof & Kazhdan) - Yes.

- canonical construction : compatible with homs of Lie bialgebras

QUE algebras

Main example - sl_2, standard structure $\begin{cases} \varphi(X^\pm) = X^\pm \wedge H \\ \varphi(H) = 0 \end{cases}$

$\rightsquigarrow U_h(sl_2)$ topological Hopf algebra gen by $X^\pm, H$. Relations $[H, X^\pm] = \pm 2X^\pm$

$$[X^+, X^-] = \frac{e^{\frac{hH}{2}} - e^{-\frac{hH}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} = \frac{\sinh(\frac{h}{2}H)}{\sinh(\frac{h}{2})}$$

$\Delta H = H \otimes 1 + 1 \otimes H$

$\Delta(X^\pm) = X^\pm \otimes e^{\frac{hH}{4}} + e^{-\frac{hH}{4}} \otimes X^\pm$. Must check compatible with $[,]$.

Antipode $S(H) = -H$, $S(X^\pm) = -e^{\pm\frac{h}{2}} X^\pm$ $S^2 \neq id$.

$\varepsilon(X^\pm) = \varepsilon(H) = 0$

- $\mathfrak{g}$, φ standard simple $/\mathbb{C}$ $\varphi(H_i) = 0$ $\varphi(X_i^\pm) = X_i^\pm \wedge H_i$

$K_i = e^{h\frac{H_i}{2}}$ $[H_i, H_j] = 0$, $[H_i, X_j^\pm] = \pm(\alpha_i, \alpha_j) X_j^\pm$

$[X_i^+, X_j^-] = \delta_{ij} \frac{K_i - K_i^{-1}}{e^{h/2} - e^{-h/2}}$ + Serre relations

Serre: $ad_{X_i^\pm}^{1-a_{ij}} X_j^\pm = \sum_{k=0}^{1-a_{ij}} \binom{n}{k} X_i^{\pm k} X_j^\pm X_i^{1-a_{ij}-k} = 0$

- deform the binomial to q-binomial $n \to [n] = \frac{e^{\frac{h}{2}n} - e^{-\frac{h}{2}n}}{e^{h/2} - e^{-h/2}}$

$= \frac{q^n - q^{-n}}{q - q^{-1}}$ $\rightsquigarrow$ q factorials, q-binomials

In Serre $\binom{n}{k} \longrightarrow \begin{bmatrix} n \\ k \end{bmatrix}_{q_i^{-2}}$ $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

In generators $E_i = X_i^+ e^{-\frac{hH_i}{4}}$ $F_i = X_i^- e^{\frac{hH_i}{4}}$

$ad_{E_i}^{1-a_{ij}}(E_j) = 0$ where $ad: A \xrightarrow{\Delta} A \otimes A \xrightarrow{id \otimes S} A \otimes A^{opp} \xrightarrow{mult} \operatorname{End} A$

$x \longmapsto x \otimes 1 + 1 \otimes x \longmapsto 1 \otimes x - 1 \otimes x$

$\varphi(a \otimes b) y = ayb$ $\rightsquigarrow$ $\varphi(1 \otimes x - 1 \otimes x) y = [x, y] \ldots$

$U_h \mathfrak{g}$ QUE alg. of $\mathfrak{g}$, complex simple fin.dim.

sl_2: $[H, X^\pm] = \pm 2X^\pm$, $[X^+, X^-] = \frac{e^{\frac{hH}{2}} - e^{-\frac{hH}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}$

Prop 1) $U_h \mathfrak{g} \overset{\varphi}{\cong} U\mathfrak{g}[[h]]$ as an algebra:

Hochschild $H^2(U\mathfrak{g}, U\mathfrak{g}) = H^2(\mathfrak{g}, U\mathfrak{g}) = 0$ for $\mathfrak{g}$ simple.

2) φ can be chosen in such a way that $\varphi/_{\mathfrak{h}} = id$ . . .

3) φ induces the canonical isomorphism $Z(U_h \mathfrak{g}) \simeq Z\ [[h]]$

The category of fin dim $\mathfrak{g}$ modules, $\mathfrak{g}$-mod

$U_h \mathfrak{g}$ - mod has as objects $U_h(\mathfrak{g})$-mod which are free of finite rank over $\mathbb{C}[[h]]$.

$F: \mathfrak{g}\text{-mod} \longrightarrow U_h \mathfrak{g}\text{-mod}$ as $V \rightsquigarrow V[[h]] \longrightarrow$ a $U_h \mathfrak{g}$ module via φ. Conversely $G: U_h \mathfrak{g} \to \mathfrak{g}\text{-mod}$, $M \longmapsto M/hM$

Prop F, G give rise to equivalences of categories

Prop For L a simple $U_h \mathfrak{g}$-mod $\Rightarrow \exists$ a dominant (for $\mathfrak{g}$) weight $\Lambda \in P_+$ and $! v_\Lambda \in L$ s.t. $X_i^+ v_\Lambda = 0$,

$H_i v_\Lambda = \Lambda(H_i) v_\Lambda$ for $1 \le i \le n$. , v_Λ generates L.

$\Rightarrow$ $L = L(\Lambda)$ highest weight simple module.

Example $U_h sl_2$ $m = 0, 1, \ldots \Rightarrow V_m$ simple $(m+1)$ dim module,

basis $\{e_k\}_0^m$ $q = e^{h/2}$. $X^+ e_k = \frac{q^{k-m} - q^{-k+m}}{q - q^{-1}} e_{k+1}$

$X^- e_k = \frac{q^k - q^{-k}}{q - q^{-1}} e_{k-1}$, $H e_k = (-2k + m) e_k$.

Uniqueness of U_h : up to change of $h \longmapsto h + \sum c_k h^k$ & choice of Cartan : cocommutative Hopf algebra C in the quantization.

Types of QUE algebras (or of Hopf algebras):

0. almost cocommutative Hopf algebra
1. coboundary H.A.
2. Triangular H.A.
3. Φ-triangular H.A. $\vdots$

Def (A,Δ) is called a.c.c. (almost cocommutative) if $\exists R \in A \otimes A$ which is invertible, and $\Delta'(a) = R\Delta(a)R^{-1}$ $a \in A$,
$\Delta' = \sigma \circ \Delta$, $\sigma(a \otimes b) = b \otimes a$ reverse comultiplication.
Ex. $(U\mathfrak{g}, \Delta_0)$ is a.c.c : $\Delta' = \Delta$, take $R = 1 \otimes 1$.

Def An acc Hopf algebra (A, R) is called quasi-triangular if $(\Delta \otimes id)(R) = R^{13}R^{23}$, $(id \otimes \Delta)(R) = R^{13}R^{12}$

Prop If (A,R) is a q.t. H.A. then
i) QYBE holds : $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ in $A \otimes A \otimes A$
ii) The linear map $A^* \to A$, $l \mapsto (l \otimes id)(R)$ is a homomorphism of algebras (in fact anti-hom of Hopf algebras).

Drinfeld's Double Construction A H.A./k., $\mathcal{D}(A) = A \otimes A^*$ as vector space. Require i) $A \hookrightarrow \mathcal{D}(A)$ is a homomorphism of Hopf algebras ii) $A^* \hookrightarrow \mathcal{D}(A)$ is an antihom of Hopf algebras
iii) we require that the canonical element $\bar{R} = (\sum e_\alpha \otimes e^\alpha)$ defines a q-T structure on $\mathcal{D}(A)$. $= \sum (e_\alpha \otimes 1) \otimes (1 \otimes e^\alpha)$

Then it is possible to define HA structure on $\mathcal{D}(A)$ with above restrictions
- QYBE easy, acc. hard

(A,R) q-t. QUE algebra. $A/hA = U\mathfrak{g}$. $\frac{R-1}{h} \bmod h = r$
gives q-t. Lie bialgebra $(\mathfrak{g}, r)$

Theorem $\mathcal{D}(U_h b_+) \cong U_h \mathfrak{g} \otimes U h[[h]]$
(infinite dim case of double construction)

Corollary $U_h \mathfrak{g}$ is a q-t. Hopf algebra
$\Rightarrow$ has R matrix

$\boxed{\text{Warning}}$ Tensor product of Hopf algebras is not always Hopf.

Ex. $\mathfrak{g} = sl_2$, $E = X^+ q^{-\frac{H}{2}}$, $F = X^- q^{\frac{H}{2}}$ $q = e^{\frac{h}{2}}$
Universal R-matrix $R = \exp_{q^{-2}}((1-q^{-2}) E \otimes F)\, q^{\frac{H \otimes H}{2}}$

where $\exp_t(x) = \sum_{n \geq 0} \frac{x^n}{(n)_t!}$ $(n)_t = \frac{t^n - 1}{t-1}$
t-exponential function.

$\mathfrak{g}$-simple $/(\mathbb{C}, (,))$ $\Rightarrow$ $U_h\mathfrak{g}$ is a quasi-triangular Hopf algebra

$\Rightarrow$ QYBE without parameters : $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$

- not shifted (like Varchenko).

Quantum group = Hopf algebra, with antipode invertible

e.g. $U_h sl_2$, $q = e^{\frac{h}{2}}$ $E = X^+ q^{-\frac{H}{2}}$, $F = X^- q^{\frac{H}{2}}$

$R = \exp_{q^{-2}}((1-q^{-2})E\otimes F)\, q^{\frac{H\otimes H}{2}}$

$U_h\mathfrak{g}$ vs. $U_q\mathfrak{g}$:

$U_h\mathfrak{g} = \{X_i^+, X_i^-, H_i\}$, introduce $K_i = e^{\frac{h}{2}H_i}$

- consider Hopf subalgebra $/\mathbb{C}$ gen. by $\{X_i^\pm, K_i^{\pm 1}\} = U_q\mathfrak{g}$, algebra over $\mathbb{C}(q)$.

(A, Δ) coalgebra $\to$ tensor product of modules $M\otimes N$, $a\cdot(m\otimes n) = \Delta(a)(m\otimes n)$ which is associative (coassoc. of Δ, $(\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta$).

(A, R) almost cocommutative, $R_{M,N} : M\otimes N \to M\otimes N$.

$C_{M,N} = P^{M,N} R_{M,N} : M\otimes N \to N\otimes M$ $\quad$ $P^{M,N}$ permute factors.

a. co-comm $\Rightarrow$ $C_{M,N}$ is isomorphism of A-modules

<u>Proposition</u> If (A,R) is quasitriangular then for any $M, N, P \in A$-mod

$C_{M,N\otimes P} = C_{M,P} \circ C_{M,N}$, $C_{M\otimes N,P} = C_{M,P}\circ C_{N,P}$

<u>Corollary</u> $C_{M,N}$ defines a braiding (= commutativity constraint) on A-mod

Monoidal category : $\mathcal{C}$, $\otimes$ s.t. $\otimes$ is associative (+pentagon axiom) & has $\mathbb{1} \in Ob\,\mathcal{C}$ s.t. $\mathbb{1}\otimes X \simeq X \simeq X\otimes\mathbb{1}$

$c_{M,N} : M\otimes N \simeq N\otimes M$ + hexagon $\Rightarrow$ braided monoidal category

<u>Strict</u> monoidal category : the associativity map $M\otimes(N\otimes P) \xrightarrow{\sim} (M\otimes N)\otimes P$ is the identity map.

$U_h\mathfrak{g}$-mod is braided monoidal as is $U_q\mathfrak{g}$ -

<u>BUT</u> $U_q\mathfrak{g}$ not q-t Hopf algebra !

$R = \exp_{q^{-2}}(E\otimes F)\, e^{\frac{h}{4}H\otimes H} \notin U_q\mathfrak{g}^{\otimes 2}$ even completed,

but it acts on any ~~finite~~ tensor of reps of $U_q(\mathfrak{g})$)

- acts as number on high weight..

Now can put $\exp_{q^{-2}}(E\otimes F)$ in usual-type completion, but $e^{\frac{h}{4}H\otimes H}$ is more tricky - we have no h, only q...

- can define on level of reps, but then harder to show hexagon...

Proposition If (A,R) is an almost cocomm Hopf Algebra $\Rightarrow$
$S^2(x) = u x u^{-1}$ for some $u \in A$, all x (S = antipode).
(Cocommutative case: $u = 1$ .. invisible).

Exercise If $R = \sum a_i \otimes b_i$ then $u = \sum_i S(b_i) a_i$ works.

Consequence: In $U_h \mathfrak{g}$ $S^2(X_i^\pm) = q_i^{\pm 2} X_i^\pm$, $S^2(H_i) = H_i$
$q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

$\Longrightarrow$ $S^2(x) = e^{h\rho} x e^{-h\rho}$.

$\rho = \sum_1^n \omega_i \in \mathfrak{h}^* = \mathfrak{h}$ sum of dominant weights $\omega_i(H_j) = \delta_{ij}$
$= \frac{1}{2} \sum_{\alpha > 0} \alpha$ in fin dim case – $\frac{1}{2}$ sum of simple roots.

$e^{h\rho} X_i^+ e^{-h\rho} = e^{h(\rho, \alpha_i)} X_i^+ = q^2 X_i^\pm$

$\Longrightarrow$ for $e^{-h\rho} u = q^{-C} \in$ Center of $U_h \mathfrak{g}$,
C = quadratic Casimir

Prop Let $L(\Lambda)$ be a simple $U_h\mathfrak{g}$-module with highest weight Λ.
Then $q^{-C}|_{L(\Lambda)}$ is a scalar operator w/ value $q^{-(\Lambda, \Lambda + 2\rho)}$
($\frac{=}{q}$ value of C on $L(\Lambda)$).

Ex. $\Delta(q^{-C}) = R^{21} R (q^{-C} \otimes q^{-C})$
R^{21} = reversed R

(Note $U\mathfrak{g}[[h]] \simeq U_h(\mathfrak{g})$ taking center to center..)
– useful to write Plücker relations: orbit Gv_Λ in $L(\Lambda)$
can be written by quadratic relations using $\Delta(c)$:
decompose $L(\Lambda)^{\otimes 2} = L(2\Lambda) + \dots$
Apply casimir : $\Delta(c)_{L^{\otimes 2}} = C_{2\Lambda} \dots$
$\Longrightarrow g(v_\Lambda \otimes v_\Lambda) = gv_\Lambda \otimes gv_\Lambda = g\, v_{2\Lambda}$, project on $L(2\Lambda)$.
$\to$ quadratic eqn relating $\Delta(c)_\Lambda$, $L(2\Lambda)$)..

Balanced category – braided monoidal with b auto. of unit functor,
$b_X : X \xrightarrow{\sim} X$ $\quad b_{X \otimes Y} = c_{XY} c_{YX} (b_X \otimes b_Y)$
— comes from quantum Casimir q^C: $b_X = q^{-C}|_X$.
our Ex. for $\Delta(q^{-C})$ gives balancing structure!

Peter-Weyl: G simple $/\mathbb{C}$ $\Rightarrow$ $\mathbb{C}[G] \simeq \bigoplus_{\text{Rep}} L^*(\lambda) \otimes L(\lambda)$

algebra of matrix elements of fin-dim $\mathfrak{g}$-modules:

ρ a rep, $l \in V^*$, $v \in V$, $a \in U(\mathfrak{g})$ $\rightarrow$ $l(\rho(a)\cdot v)$

matrix elements form algebra. ~ linear functionals of $U(\mathfrak{g})$, so can identify with subalgebra of $(U\mathfrak{g})^*$ dual Hopf algebra.

Fix q generic, $\Rightarrow U_q\mathfrak{g}$.

Def $\mathbb{C}[G]_q$ is the algebra of matrix elements of fin-dim reps of $U_q\mathfrak{g}$ — or rather only admissible reps:

$V = \bigoplus_{\lambda \in X^*} V_\lambda$, $K_i V_\lambda = q^{(\lambda, \alpha_i)} V_\lambda$. $\Longrightarrow$ restricted dual + admissibility.

— to rule out extra "parasitic" reps .. not a problem for $U_h\mathfrak{g}$ but is for $U_q\mathfrak{g}$ ..

Assume $q \in \mathbb{R}$ $\longrightarrow$ define "algebra of functions on maximal compact" $\mathbb{C}[K]_q$: in terms of compact (Cartan) involution:

$(\mathbb{C}[K], \bar{\ }) \simeq (\mathbb{C}[G], *)$, $*$ is dual to involution

$\theta : U\mathfrak{g} \to U\mathfrak{g}$, $\theta : X_i^{\pm} \mapsto X_i^{\mp}$, $H_i \mapsto H_i$.

Ex. $\mathbb{C}[SL_2\mathbb{C}]$: $\det\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22}\end{pmatrix} = 1$. $t_{11}^* = t_{22}$, $t_{12}^* = -t_{21}$

$\Leftrightarrow (\mathbb{C}(SU_2), \bar{\ })$.

$\Rightarrow$ Def $\mathbb{C}[K]_q = (\mathbb{C}[G]_q, *)$

e.g. $\mathbb{C}[SL_2\mathbb{C}]_q$: $t_{11}t_{21} = q t_{21} t_{11}$, $t_{21}t_{12} = t_{12}t_{21}$ etc.

q-commute, commute, $[t_{22}, t_{11}] = (q-q^{-1}) t_{12} t_{21}$,

$\det_q = t_{11}t_{22} - q t_{12} t_{21} = 1$.

$*$: $t_{11}^* = t_{22}$, $t_{12}^* = -q t_{21}$

$\Rightarrow$ $q\text{-det} = t_{11}^* t_{11} + t_{12}^* t_{22}$ "=" $|t_{11}|^2 + |t_{22}|^2 = 1$.

$q=1$ $\mathbb{C}[K]$ commutative, so reps are characters $\Leftrightarrow$ points of K.

What is rep theory of $\mathbb{C}[K]_q$?

J ideal of $A / \mathbb{C}[[h]]$ $\longrightarrow$ J_0 is Poisson ideal

$\Rightarrow$ subvariety of Spec A_0

Primitive ideal $\Rightarrow$ minimal Poisson subvariety, i.e. symplectic leaves of A_0 : should have relation between irreps & symplectic leaves.

Rep : $\pi : \mathbb{C}[\cancel{K}]_q \longrightarrow \mathrm{End}\ H$, $\pi(a^*) = \pi(a)^*$

— rep as algebra

1. $\mathbb{C}[SU_2]_q$ Theorem There are 2 families of irreps of $\mathbb{C}[SU_2]_q$:

a. 1-dim, $t \in S^1$, $\tau_t(t_{11}) = t$, $\tau_t(t_{21}) = 0$.

b. ∞-dim irreps π_t, $t \in S^1$: Fix unitary basis $\{e_k\}_{k\geq 0} \subset H$.

$\pi_t(t_{11})\, e_k = e_{k-1} \cdot (1-q^k)^{\frac{1}{2}}$

$\pi_t(t_{21})\, e_k = t \cdot q^k\, e_k$: t_{21} diagonal, t_{11} : $e_0 \mapsto 0$

Thus $t_{11} \rightsquigarrow X^+$, $t_{21} \rightsquigarrow H$: looks like b of fin dim reps.

General case $\mathbb{C}[K]_q$:

2a. Highest weight approach: $A_+ = \{ l(P_\Lambda(a) V_\Lambda) \}$

$P_\Lambda : U_q\mathfrak{g} \to \mathrm{End}\ L(\Lambda)$ high weight rep, V_Λ high weight vector .. $A_+ \longleftrightarrow$ "Borel" $U(b_+)$.

- any irrep there's a unique line invariant wrt A_+.

2b. Fix i - vertex of Dynkin diagram of $\mathfrak{g}$. $\Rightarrow$ embedding of Hopf algebras $\varphi_i : U_q(sl_2) \hookrightarrow U_q(\mathfrak{g})$

$\varphi_i^* : \mathbb{C}[G]_q \longrightarrow\!\!\!\!\rightarrow \mathbb{C}[SL_2\mathbb{C}]_q$ compatible with involutions : so may replace $G \to K$, $SL_2 \to SU_2$.

Denote π_{-1} the rep π_t of SU_2 for $t = -1$.

$\Rightarrow$ $\pi_{-1} \circ \varphi_i^*$ irrep of K, denote by π_i

Fix $w \in W$ Weyl group, $w = s_{i_1} \cdots s_{i_k}$ reduced expression $\Rightarrow$ rep $\pi_{i_1} \otimes \cdots \otimes \pi_{i_k}$

Theorem This tensor product is irreducible and depends on w only. $\rightsquigarrow \pi_w$.

Theorem a. $\{1\text{-dim reps } \tau_t \text{ of } \mathbb{C}[K]_q\} \xleftrightarrow{1\text{-}1 \text{ corresp}} \{t \in T \text{ maximal torus of } K\}$

b. Any irrep of $\mathbb{C}[K]_q$ is isomorphic to $\pi_w \otimes \tau_t$ for some w, t.

Thus irreps of $\mathbb{C}[K]_q$ correspond exactly to symplectic leaves of K ..
π rep => Ker π ideal, Ker π (mod h) gives Poisson ideal in $\mathbb{C}[K]$, gives closure of unique symplectic leaves.
$\Sigma_w = \Sigma_{i_1} \cdots \Sigma_{i_k}$ Schubert cells decomposition of orbit gives <-> construction $\pi_w = \pi_{i_1} \otimes \cdots \otimes \pi_{i_k}$ irreps

Weyl group: $W = N(T)/T$: can choose subgroup $\bar{W} \subset K$ of representatives of W.
- put δ-function at each $w \in W$, δ_w => 1-dim rep $f \mapsto f(w)$ assoc to w.

In our Th have vector $\xi_0 \otimes \cdots \otimes \xi_0 = e_w$ distinguished vector. Now define $\bar{w} : \mathbb{C}[K]_q \to \mathbb{C}$ as $\bar{w}(f) = (\pi_w(f) e_w, e_w)$ linear function on $\mathbb{C}[K]_q$
- from which we can reconstruct $\bar{w} \subset K$ in semiclassical limit.

Theorem $\bar{s}_i \bar{s}_j \bar{s}_i \cdots = \bar{s}_j \bar{s}_i \bar{s}_j \cdots$ (braid/Coxeter relations), s_i simple reflections.

(Note $\bar{W} \subset K$ not canonical, projects on to W with kernel with various $\mathbb{Z}/2$ .. but once we pick $\bar{W} \subset SU_2$ it's canonical for all other groups).

Set $T_i : U_q\mathfrak{g} \to U_q\mathfrak{g}$, $T_i x = \bar{s}_i x \bar{s}_i^{-1} \in \mathbb{C}[K]_q^*$ but in fact $T_i x \in U_q\mathfrak{g}$ .. <-> Weyl group acts on $U_q\mathfrak{g}$.

. In classical case may define ~~positi~~ roots as orbits under W of simple roots, $E_\alpha = w E_i w^{-1}$ $w\alpha_i = \alpha$
=> q-root vectors $E_\alpha = T_w(E_i)$
$\bar{s}_i^2$ is no longer 1 or anything simple, just some element in $U_q\mathfrak{g}$.

Def q-Weyl gp is the Hopf algebra gen. by $U_q\mathfrak{g}$ and all $\bar{w}$, $w \in W$: analog of semidirect of W, $U\mathfrak{g}$.

Theorem Universal R-matrix (ADE) $R = \prod_{\alpha > 0} \exp_{q^{-2}}\left((1-q^{-2}) E_\alpha \otimes F_\alpha\right) q^{t_0}$
$t_0 \in \mathfrak{h} \otimes \mathfrak{h}$ canonical

Take $\bar{w}_0$ quantization of longest W element $\Delta(\bar{w}_0) = R^{-1}(\bar{w}_0 \otimes \bar{w}_0)$
almost group like .. rather $\bar{w}_0' = \bar{w}_0 q^{-\frac{1}{2}\sum I_k^2}$ I_k ortho basis.