Toby Stafford: NC ruled surfaces

Note Title 5/15/2007

$$A = \mathbb{C}\langle x, \partial \mid \partial x - x\partial - 1 \rangle$$
$$= \mathcal{D}(\mathbb{A}^1) \quad \text{Weyl algebra}$$

Noetherian domain, hereditary:
all right ideals are projective,
simple (no 2-sided ideals)

$K^0(A_1) = \mathbb{Z}$ & all projective
A_1-modules of rank ≥ 2 are free
<u>but</u> $\exists$ lots of non-cyclic right ideals

- $xA_1 + \partial A_1 = A_1$

$$\Rightarrow \quad xA_1 + \partial^m A_1 = A_1 \quad \text{by induction}$$

$$\begin{array}{c} xA_1 + \partial^m A_1 = A_1 \\ \uparrow \phi \\ A_1 \oplus A_1 \\ \cup \\ P_m = \ker \phi = \{ r : \partial^m r \in xA_1 \} \end{array}$$

$$P_m = x^{m+1} A_1 + (x\partial - m) A_1$$

This is a proper ideal. If it were cyclic would need generator $\mid x^{m+1}$

$\Rightarrow$ it's x, but also $\mid x\partial - m \ldots$

$\Rightarrow$ not cyclic.

<u>Nb</u> $\mathrm{End}_{A_1}(P_m) = \mathcal{D}(C_m)$

$C_m =$ cusp $u^2 = v^{m+2}$

<u>Problem</u> Classify all right ideals of A_1

Answer: Cannings-Holland mid 90s
Berest-Wilson (CM spaces
Le Bruyn (projective geometry)
Kapustin-Kuznetsov-Orlov
Nevins-Stafford (Sensitive moduli)
Berest-Chalykh (A_∞)
BZ - N
...

Ans: $\exists$ an integer invariant c_2 (2nd Chern class) so that right ideals with $c_2 = n$ $\longleftrightarrow$ n^{th} Calogero-Moser space

$$\mathcal{C}_n = \left\{ X, Y \text{ } n\times n \text{ matrices} : [X,Y] - I \text{ rank } 1 \right\} / GL_n$$

$$\left\{ \begin{array}{l} \text{all right} \\ \text{ideals up to ism} \end{array} \right\} = \coprod \mathcal{C}_n$$

Note that A_1 is the coord ring of the NC affine plane

We filter A_1 by either $\Gamma_n^\circ = \mathcal{D}_{\leq n}$ or Bernstein filtration $\Gamma_n = \mathbb{C}\langle x, \partial\rangle_{\leq n}$ total degree

$$gr_\Gamma A_1 = \mathbb{C}[x,y] = \mathcal{O}(T^*\mathbb{C})$$

To get a good moduli space we need to complete A_1 to get an NC projective space

$S = \mathbb{C}\langle x, \partial, z \mid \partial x - x\partial - z^2;\ z \text{ central}\rangle$

$\simeq$ Rees ring $= \bigoplus \Gamma_n t^n$

generated by $t = z$, $xt = x'$, $\partial t = \partial'$

The "line at ∞" $\quad S/zS = \mathbb{C}[\bar{\partial}, \bar{x}]$

$\mathrm{Proj}\,(S/zS) = \mathbb{P}^1$.

"Proj S" $\longleftrightarrow$ $\mathbb{P}^2_{NC}$

$\mathrm{gr}\, S$ = all f.g. graded modules

$\mathrm{qgr}\, S = \mathrm{gr}\, S / \text{fin dim submodules}$

$:= \mathrm{coh}\ \mathbb{P}^2_{NC} = \overline{T^*_{\hbar}\,\mathbb{C}}$

[Serre: if C is a commutative connected graded domain, generated in deg 1

$\Rightarrow \mathrm{coh}\ \mathrm{Proj}\, C = \mathrm{qgr}\, C$]

$\mathrm{qgr}\, \bar{S} = \mathrm{coh}(\mathbb{P}^1_\infty) \subset \mathrm{coh}(\mathbb{P}^2_{NC})$

Funny thing: complement of this line is the Weyl algebra, which is hereditary <---> "1-dimensional"

Given a right ideal M of A_1, want to construct a graded module /S

Def A torsion-free filtration on M is a filtration $M = \bigcup_{i \geq 0} \Lambda_i$ (regarding Bergh.)

s.t. $gr_\Lambda M = \bigoplus \Lambda_i / \Lambda_{i-1}$ is torsion-free in coh ($\mathbb{P}^1_\infty$) [ie $gr_\Lambda M$ may have a fin dim submodule but eventually factors are torsion free].

Lemma (Berest-Wilson)

1. Up to a shift such a filtration is unique: identify $M = I \subset A_1$ right ideal

$\exists!\, r$ s.t. $\Lambda_n = I \cap \Gamma_{n+r}$ $\forall n \gg 0$

2. So $\exists$ a ! torsion-free filtration on M st. $gr_\Lambda M = \mathcal{O}_{\mathbb{P}^1_\infty}$

3 Given M & a t-f filtration
$\Rightarrow \mathcal{R}(M) = \mathcal{M} = \bigoplus \Lambda_n z^n$
gives me a torsion-free rank 1 module in qgr-S

Corollary $\mathcal{M} = \mathcal{R}(M)$ is a framed torsion free sheaf on S:
$$\mathcal{M}/z\mathcal{M} = \mathcal{O}_{\mathbb{P}^1_\infty} \quad \text{in coh } \mathbb{P}^1_\infty$$

$\Rightarrow$ rk 1 projective A_1-modules $\longleftrightarrow$ framed t-f sheaf of rank 1 on $\mathbb{P}^2_{NC}$.

$[\ M = \mathcal{M}[z^{-1}]_0\ ]$

Classifying rt ideals of A_1 $\Longleftrightarrow$ classifying framed t-free sheaves on coh $\mathbb{P}^2_{NC}$
(Le Bruyn, KKO, NS, BN, ...)

Technical lemma: The analogues of $H^m(\mathbb{P}^2, \mathcal{M})$ are

$$H^m(\mathbb{P}^2_{NC}, \mathcal{M}) = \mathrm{Ext}^m_{\mathrm{qgr}\text{-}S}(S, \mathcal{M})$$

If $\mathcal{M}$ is framed, $H^0(\mathcal{M}(j)) = H^2(\mathcal{M}(j)) = 0$
$j = 1, 2$

$\mathcal{M}(r) = \bigoplus \mathcal{M}(r)_t$
where $\mathcal{M}(r)_t = \mathcal{M}_{r+t}$.

$$\dim H^1(\mathbb{P}^2_{NC}, \mathcal{M}(-2)) = n$$
$$= \dim H^1(\mathbb{P}^2_{NC}, \mathcal{M}(-1))$$
$$= 1 + \dim H^1(\mathbb{P}^2_{NC}\, \mathcal{M})$$
$= 2^{nd}$ Chern class $= 1 -$ Euler characteristic

$\Rightarrow$ get a quiver

$$H^1(\mathcal{M}(-2)) \underset{x,y,z}{\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}} H^1(\mathcal{M}(-1)) \underset{x,y,z}{\overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}} H^1(\mathcal{M})$$

$(Y = \partial)$ satisfying $YX - XY - Z^2 = 0$ $(*)$

<u>Technical lemma</u> The first Z map is an isomorphism $H^1(\mathcal{M}(-2)) \simeq H^1(\mathcal{M}(-1)) =: V$ & second Z map is a surjection

$\Rightarrow$ (*) gives

$YX - XY - I$ has rank one as a map $V \to V$.

$\Rightarrow$ <u>Theorem</u> The framed t-free rank 1 modules on $\mathbb{P}^2_{NC}$ with $c_2 = n$ are parametrized by $\mathcal{C}_n = \{ X, Y \in M_{n \times n}(\mathbb{C}) : \underset{\text{rank } 1}{XY - YX - I} \} / GL_n$

e.g. $P_{n-1} = x^n A_1 + (x\partial - (n-1)) A_1$

$V = \mathbb{C}^n$ basis $1, x, \dots, x^{n-1}$

x, ∂ acting in natural way

$$x \longleftrightarrow \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & 1 & \ddots & \\ & & 1 & \end{pmatrix} \qquad \partial \longleftrightarrow \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & n-1 \\ & & & 0 \end{pmatrix}$$

$$\partial x - x\partial - I = \begin{pmatrix} 0 & 0 \\ 0 & -(n-1) \end{pmatrix}$$