S. Stolz: Elliptic Cohomology (w P. Teichner) 8/9/04

<u>Witten Genus</u> M oriented manifold

$\varphi_W(M) \in \mathbb{C}[[q]]$

E vector bundle $\Rightarrow S_q E = \underline{1} + E\cdot q + S^2E\cdot q^2 + \dots$

$$\varphi_W(M) = \int_M ch\left[\bigotimes_{k=1}^{\infty} S_{q^k}\left((TM - \underline{\mathbb{R}}^n)\otimes\mathbb{C}\right)\right]\hat{A}(TM)$$

$$= \int_M \hat{A}(TM) + q\int_M \hat{A}(TM)\, ch\left((TM-\underline{\mathbb{R}}^n)\otimes\mathbb{C}\right) + \dots$$

all indices of twisted Dirac operators

M	Structure gp	topological condition	φ_W
oriented	$SO(n)$	$w_1=0$	defined
spin	$Spin(n)$ $\uparrow$	$w_1=w_2=0$	coefficients $\in\mathbb{Z}$
string	$String(n)$ $\uparrow$ (kill π_3) = 3-connected cover of $O(n)$	$w_1=w_2=0, \frac{p_1}{2}=0$	wt $\frac{n}{2}$ modular form MF_n (degree n)

Family version: $\begin{matrix} M \\ \downarrow \\ X\end{matrix}$ family of string manifolds $/X$

<u>Homotopical Construction</u> (Hopkins, Miller, Ando, Strickland, Rezk)

$\varphi_W(M/X) \in TMF^{-n}(X)$ topological modular forms

$TMF^{-n}(pt)\otimes\mathbb{C} \cong MF_n$ modular forms for $SL_2\mathbb{Z}$

[connective version would give ones holomorphic at ∞,
periodic version - invert discriminant]

$\varphi_W(M/pt)$ is original Witten genus

CFT Witten: for suitable M^n, $\varphi_W(M)$ = partition function of a CFT

Goal: give a construction of $\varphi_W(M/X)$ via CFTs

<u>Cohomology Theories</u> $\{E_n\}_{n\in\mathbb{Z}}$ $\Omega E_{n+1} \sim E_n$ homotopy
Ω spectrum

$\Rightarrow$ associate cohomology theory: $E^n(X) := [X, E_n]$

Ex. $KO^{-n}(pt) = \begin{cases} \mathbb{Z} & n \equiv 0\ (4) \\ \mathbb{Z}/2 & n \equiv 1,2\ (8) \\ 0 & \text{otherwise} \end{cases}$

Warmup example: construct family version of $\hat{A}$ genus $\hat{A}(M/X)$, M/X spin in $KO^{-n}(X)$, in field theoretic terms.

M spin $\Rightarrow \hat{A}(M) = \text{index}(\cancel{D})$ ...

Superparticle in M $\quad \mathbb{R}^{1|1} \xrightarrow[x]{} M$

$\Rightarrow$ quantize to $\mathcal{H}$: L^2-spinors on M.

Symmetry: translation action of $\mathbb{R}^{1|1}$ (classically)

$\Rightarrow$ quantize $(t,\theta) \longmapsto e^{-tD^2+\theta D}$

$\mathbb{R}^{1|1}_+ \longrightarrow$ Hilbert Schmidt $(\mathcal{H})$

$\mathbb{R}^{1|1}_+$ = moduli of superintervals = metrics on standard superinterval

$\Rightarrow$ Euclidean 1-dimensional field theory (SUSY)

$$\hat{A}(M) = \mathrm{str}(e^{-tD^2+\theta D})$$

Refinement: use Clifford-linear spinor bundle
$S = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} C_n$ full Clifford algebra

$\Rightarrow L^2(M,S)$ has right C_n action $\cancel{D}$ C_n linear

$\Rightarrow$ 1 Euclidean field theory of degree n.
SUSY

EFT_{-n} = Euclidean SUSY field theories of degree n

__Theorem__ 1. $\{EFT_{-n}\}$ form an Ω-spectrum

2. $[X, EFT_{-n}] \cong KO^{-n}(X)$

(an EFT: Hilbert space $H \circlearrowleft C_n$ & semigroup homomorphism $\mathbb{R}_+^{1|1} \longrightarrow HS(H)$)

Family index: $\begin{matrix} M \\ \downarrow \\ X \end{matrix}$ spin $\Rightarrow X \longrightarrow EFT_{-n}$

$x \longmapsto$ EFT of fiber M_x.

$$\hat{A}(M/pt) = \begin{cases} \hat{A}(M) & n \equiv 0 \ (8) \\ \tfrac{1}{2}\hat{A}(M) & n \equiv 4 \ (8) \end{cases}$$

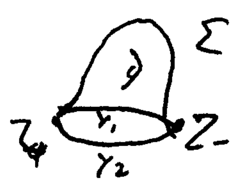
__Hope__ M string $\rightsquigarrow \exists$ CFT U with partition function $\varphi_w(M)$, SUSY, 3-tier, degree n

$Z^{0|1}$

$Y^{1|1}$

i.e. $Z^0 \rightsquigarrow U(Z^0)$ von Neumann algebra

$Z^+ \overset{Y'}{\frown} Z^- \rightsquigarrow U(Y)$ $U(Z_+)-U(Z_-)$ bimodule



$\rightsquigarrow U(\Sigma, \varphi): U(Y_1) \to U(Y_2)$ map of bimodules

$\Sigma^{2|1}$

$\varphi \in$ Pfaffian line of Spin RS

(0,1)-SUSY ("$N = \frac{1}{2}$")

$\varphi \in F(\Sigma)^{\otimes n}$ $\quad F(\Sigma) = \begin{cases} Pf(\Sigma) & \Sigma \text{ closed} \\ \Lambda^\bullet L, \ \partial\Sigma \neq \emptyset, & \Sigma \text{ connected} \end{cases}$

L = ∂-values of harmonic spinors on Σ

$\Lambda^\bullet L$ = Fock space

+ gluing axioms

CFT_n = {space of CFTs of degree $-n$}

Partition function of U: 

$\left(\underset{\substack{\cap \\ Pf(T_\tau)^{\otimes n}}}{\psi} \longmapsto \underset{\substack{\cap \\ \mathbb{C}}}{U(T_\tau,\psi)}\right) \in Pf(T_\tau)^{\otimes n}$

$\Rightarrow$ modular form of degree n
(RR spin structure)

<u>Example</u> (degree n): $U(pt) = \overline{\mathrm{Cliff}(L^2(I,\mathbb{R}^n))} = A^{\otimes n}$

A vN algebra assoc to $\mathbb{R}$, III_1-factor

$U(\Sigma,\psi) \in F(\Sigma)^{-n}$: $\psi \otimes \Omega \in \overline{F(\Sigma)}^{\otimes n} \otimes F(\Sigma)^{\otimes n}$
$= U(\partial\Sigma)$

Ex. $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ n-dim vector bundle + string structure $\Rightarrow$ map $X \to CFT_n$

<u>Conjecture A</u> Can extend

$$\begin{tikzcd} X \arrow[dr, "\text{Euler class}"'] & \subseteq & E \cup \{\infty\} \arrow[dl, dashed, "\text{Thom class}"] \\ & CFT_n & \end{tikzcd}$$

If true get $S^1 = \mathbb{R} \cup \{\infty\} \xrightarrow{s} CFT_1$

$S^1 \wedge CFT_n \xrightarrow{s \wedge id} CFT_1 \wedge CFT_n \xrightarrow{\otimes} CFT_{n+1}$

$\Leftrightarrow$ $CFT_n \longrightarrow \Omega CFT_{n+1}$

<u>Conjecture B</u> This is a homotopy equivalence, making $\{CFT_n\}$ an Ω spectrum, $CFT^{-n}(X) = [X, CFT_{-n}]$

Question! is $CFT_n \sim TMF_n$?

Consequence: $\begin{array}{c} M \\ \downarrow \\ X \end{array}$ string-n family $\Rightarrow$ $CFT^{-n}(X) = [X, CFT_{-n}]$

$\Rightarrow$ Note only see homotopy theory of spaces CFT_n.