S. Stolz (w. P. Teichner) — Spinor bundles on loop spaces — 2/11/03

Motivation: M manifold, $E^n \to M$ real vector bundle with spin structure $\Rightarrow$ $Spin_n(E) \to M$ principal $Spin_n$ bundle, associate Clifford spinor bundle $S(E)$ to C_n = Clifford algebra on $\mathbb{R}^n$

- $S(E)$ represents the Euler class of E in $KO^n(M)$ (orientability condition for having Euler class in KO is being spin)
- $L^2(M, S(TM)) \circlearrowleft$ Dirac operator

E vector bundle with string structure ($\Rightarrow$ $w_1 = w_2 = 0$ & $\lambda(E) \in H^4(M_{\mathbb{Z}})$, $2\lambda(E) = p_1(E)$) $\Rightarrow$ $\mathcal{F}(E)$ Fock space bundle on LM $\rightsquigarrow$ version of spinor bundle on LM. $\overset{||}{0}$

Hope:
- $\mathcal{F}(E)$ represents Euler class of E in $TMF^n(M)$ topological modular forms (Hopkins-Miller). $\mathbb{Q} \otimes TMF^*(pt)$ = ring of modular forms $[\Delta^{-1}]$
- "Dirac-Witten" operator acting on sections of $\mathcal{F}(TM)$ (S^1-equivariant index of this should be Witten genus)

Spin structures on inner product spaces V (dim n), $C(V)$ = Clifford algebra on V

.... a spin structure on $C(V)$ is a $C(V)$-$C(\mathbb{R}^n)$ bimodule B_V, irreducible & $\mathbb{Z}/2$-graded.

- two isomorphism classes of such, $\longleftrightarrow$ orientations of V (categorification of orientations: giving an honest bimodule)

Morphism from $(V, B_V) \longrightarrow (W, B_W)$: isometry $f: V \longrightarrow W$ + isomorphism $B_V \xrightarrow[f]{\cong} B_W$ of $C(V)$-$C(\mathbb{R}^n)$ bimodules

- A certain von Neumann algebra:

S = Clifford spinors on S^1, $H = L^2(S^1; S) \circlearrowleft \mathbb{C} = Cl(\mathbb{R})$

α grading involution, anti-linear.

$b(v,w) := \langle \alpha(v), w \rangle$ $\quad$ $\mathbb{C}$-bilinear

$C := Cliff(H, b)$ ... get modules from Lagrangians of H: $L = \{\partial\text{-values of harmonic spinors on } B^2\} \subset H$ Lagrangian

$C \circlearrowright \Lambda^* L$ $\quad$ $\mathcal{F}$ = Fock space = completion of $\Lambda^* L$

$C \longrightarrow \mathcal{B}(\mathcal{F})$

Look at sections supported on $I \subset S^1$ upper half-circle

$\mathrm{Cliff}(L^2(I;S)) \subset C \hookrightarrow \mathcal{B}(\mathcal{F})$

A = von Neumann (weak) closure of this subalgebra.

- ~~the~~ type III_1 hyperfinite factor. (unique up to iso)

$A_n = A^{\otimes n}$ (n-component spinors)

- can define fusion using bimodules for these (Wassermann)

$$\begin{array}{ccccc} \Omega\,\mathrm{Spin}(n) & \longrightarrow & P\,\mathrm{Spin}(n) & \longrightarrow & \mathrm{Spin}(n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Inn}(A_n) & \longrightarrow & \mathrm{Aut}(A_n) & \longrightarrow & \mathrm{Out}(A_n) \end{array}$$

$P\,\mathrm{Spin}_n$ = paths $\gamma : I \to \mathrm{Spin}_n$, $\gamma(1) = \mathrm{Id}$

A string structure on V^n is a pair (A_V, α_V):

- A_V von Neumann algebra
- $\alpha_V : \mathrm{Spin}(\mathbb{R}^n, V) \longrightarrow \mathrm{Out}(A_n, A_V) = \mathrm{Isom}(A_n, A_V) / \mathrm{Inn}(A_n)$

Spin$_n$ equivariant

$K(\mathbb{Z},2) \to \mathrm{String}(\mathbb{R}^n, \mathbb{R}^n) \xrightarrow[\text{6-connected cover}]{} \mathrm{Spin}(n)$ very big group

$K(\mathbb{Z},2) = \mathrm{Inn}(A)$

$$\begin{array}{ccccc} \mathrm{Inn}(A_n) & \to & \mathrm{String}(n) & \longrightarrow & \mathrm{Spin}(n) \\ \parallel & & \downarrow & & \downarrow \\ \mathrm{Inn}(A_n) & \longrightarrow & \mathrm{Aut}(A_n) & \longrightarrow & \mathrm{Out}(A_n) \end{array}$$

Structure on $S(E)$: ~~(Gauss connection on E)~~

Connection on E = functor from path groupoid of M to Hilbert spaces

$x \longmapsto S(E)_x$, morphisms $\longleftrightarrow$ parallel translation.

$\mathcal{F}(E)$ gives functor from Ob $\{$ 1-manifold $\xrightarrow{\gamma} M$ $\} \longmapsto \mathcal{F}(E)_\gamma$ (Hilbert space)

Morph $\{$ 2-manifold w/ boundary  $\} \longmapsto$ ~~2-manifold~~ Hilbert-Schmidt operator $\mathcal{F}(E)_{\gamma_1} \to \mathcal{F}(E)_{\gamma_2}$

- need string structure on E + "geometric string structure" ~ analog of connection.

$\longrightarrow$ Segal's elliptic objects over M.

Excision $\longleftrightarrow$ need locality property for elliptic objects -

$\rightsquigarrow$ diff: $\mathcal{F}(E)$ gives a functor between 2-categories

Geodesic 2-category

Obj $\cdot\, x$

Mor $x \xrightarrow{\gamma} y$

Composition 

2-morphisms 

conformal surface

algebraic 2-category

A_x von Neumann algebra

bimodule for A_y-A_x, B_γ

$B_\gamma \otimes_{A_y} B_\gamma$ "Connes fusion"

$\mathrm{HilSchmidt}_{A_x\text{-}A_y}(B_\gamma, B_\delta)$