

B.Tsygan - (Cyclic Homology -)

1/27/03

[Introduction to NC geometry]

### I. NC calculus

First: Motivation, low-dim examples, Hochschild & cyclic cochains

1. How to describe a one-cycle on a manifold  $M$  in terms of the algebra of functions  $A = C_M, C^*(M)$  cf. ?

Given  $a, b \in A \Rightarrow$  can take  $\bar{\tau}(a, b) := \int_a b$

... any 1-form is linear combo of expressions  $a \cdot db$

$\bar{\tau} : A \otimes A \rightarrow k$  ( $= \mathbb{R}, \mathbb{C}, \dots$ )  $A$  associative/k characteristic zero

Fact: 1)  $\bar{\tau}(ab, c) - \bar{\tau}(a, bc) + \bar{\tau}(c, ab) = 0$  ( $\rightarrow$  a cocycle)  
 2)  $\bar{\tau}(a, b) + \bar{\tau}(b, a) = 0$  ( $\rightarrow$  a cycle)

Why this ordering of entries (in NC case?)

$$[1 \leftrightarrow \int a \, d(bc) - \int ab \cdot dc + \int a \, db \, c]$$

Leibniz

2  $\leftrightarrow$  integration by parts. Makes sense e.g. for

$a, b, c \in M_N(A)$  matrix-valued, with  $\bar{\tau}(a, b) = \text{Tr} \int a \, ds$

1  $\leftrightarrow$  functional on one-forms.

$$A \xrightarrow{d} \Omega_{A/k} = A \otimes A / (a \otimes b - ab \otimes 1)$$

$$a \mapsto 1 \otimes a$$

$$2 \xrightarrow{\text{vans on exact}} \bar{\tau}(1, a) = 0$$

(note  $\bar{\tau}(a, 1) = 0$  automatically by 2)

Def: A Hochschild/cyclic one-cocycle on  $A$  is  $\bar{\tau} : A^{\otimes 2} \rightarrow k$  satisfying  $\frac{d}{1+2}$ .

Observations: • a one-cocycle of  $A$  extends to  $M_N(A)$  via

$$\bar{\tau}(a, b) = \sum_{i,j} \bar{\tau}(a_i, b_j), \quad \text{also one-cocycle}$$

• a cyclic one-cocycle  $\bar{\tau}$  of  $A$  determines a Lie algebra two-cocycle of  $A$  with bracket  $ab - ba$

$\Leftrightarrow$  a Lie algebra central extension of  $\text{cyl}_N(A) = M_N(A)$ .

$$\text{i.e. } \begin{aligned} \bar{\tau}([a, b], c) + \bar{\tau}([b, c], a) + \bar{\tau}([c, a], b) &= 0 \\ \bar{\tau}(a, b) + \bar{\tau}(b, a) &= 0 \end{aligned}$$

$A \otimes \bar{A}$  (same),  $\bar{A} = A/k = 1$

A noncommutative:  $\bar{\Omega}'_{A/k} = A \otimes A / (a \otimes b - ab \otimes 1)$

neither left nor right module, but

$$A \xrightarrow{d} \bar{\Omega}'_{A/k}$$

still well defined

Rank of an idempotent  $e^2 = e \in M_N(A)$ :

$A^N e$  left projective  $A$ -module.

$$\text{let } \text{rk}(e) = \sum_{i=1}^N e_{ii} \in A.$$

But  $A^N e \cong A^M e'$  as modules won't have the same rank

but  $e = xy$ ,  $e' = yx$  for some rectangular matrices,

$$\Rightarrow \text{rk } e = \text{rk } e' \text{ mod } [A, A]$$

( $x, y$  come from composition  $A^N \rightarrow A^N e \cong A^M e' \hookrightarrow A^M$  etc.)

$\Rightarrow$  well defined  $\text{rk}(P) \in A/[A, A]$  for any projective module  $[A, A] = \text{linear span of commutators}$ .

Facts: 1.  $d(\text{rk}(e)) = 0$  in  $\bar{\Omega}_{A, k}$  ... ie  $\text{rk}$  is "locally constant"  
 2.  $d$  kills  $[A, A]$  so  $d(\text{rk}(P)) = 0$  makes sense.

Proof: 1. :  $(2e+1) \otimes e \otimes e$

$$\begin{array}{ccc} & A \otimes \bar{A} \otimes \bar{A} & a_0 \otimes g, \otimes a_n \\ & \downarrow & \text{I} \\ c_1 \otimes e & A \otimes \bar{A} & a_0 g, \otimes a_n - a_0 \otimes g, a_2 + a_2 a_0 \otimes a_1, \\ \xrightarrow{A} & & \end{array}$$

$$1 \otimes e \otimes e \rightarrow 2e \otimes e - 1 \otimes e \quad e \otimes e \otimes e \mapsto e \otimes e$$

matrix version:  $\sum_{i,j,k} (2e+1)_{ij} \otimes e_{jk} \otimes e_{ki} =: ch_e(e)$   
 kills  $1 \otimes e \otimes e$  in  $A \otimes \bar{A}$ , so set  $d(\text{rk}(e)) = 0$ .

2.

$$\begin{array}{ccc} a_0 \otimes a_1 & \xrightarrow{\quad} & 1 \otimes a_0 \otimes g, - 1 \otimes a_1 \otimes a_0 \\ \downarrow & & \downarrow \\ a_0 g, - a_1 g & \xrightarrow{\quad} & 1 \otimes (g, a_0 - a_1 a_0) \end{array}$$

□

More general  
complex:

$$\begin{array}{ccccc} & A \otimes \bar{A} \otimes \bar{A} & & a_0 a_1 \otimes a_n & \\ & \downarrow & & \downarrow & \\ A \otimes \bar{A} & \xrightarrow{\quad} & A \otimes \bar{A} & a_0 a_1, & \\ & \downarrow & & \downarrow & \\ A & \xrightarrow{\quad} & A & a_0 a_1 - a_1 a_0 & \end{array}$$

beginnings  
of a morphism  
of complexes ... try to extend!

Back in geometry:

$P$ -cycle on  $M \cdot \& g_0, \dots, g_p \in M_N(1)$  ( $A = \text{Fun}(m)$ )

$\Rightarrow$  can take  $\bar{\tau}(g_0, \dots, g_p) := \text{tr} \int_C g_0 dg_1 \dots dg_p$

$$1. \sum_{i=0}^{P+1} (-1)^i \bar{\tau}(g_0, \dots, g_i, g_{i+1}, \dots, g_{p+1}) + (-1)^{P+1} \bar{\tau}(g_{p+1}, g_0, g_1, \dots, g_p) = 0$$

$$2. \bar{\tau}(g_0, g_1, \dots, g_p) = (-1)^P \bar{\tau}(g_0, g_1, \dots, g_p)$$

$A \xleftarrow[\text{cyclic}]{} \text{Hochschild}$   $P$ -cocycle of  $A$  is  $\bar{\tau}$  satisfying  $\bar{\tau} = 0$

1)  $\leftrightarrow$  chain  $\Leftrightarrow$  2)  $\leftrightarrow$  cycle in geometry.

$$\text{Define } b(g_0 \otimes g_1 \otimes \dots \otimes g_{p+1}) = \sum_{i=0}^P (-1)^i g_0 \otimes \dots \otimes g_i \otimes g_{i+1} \otimes \dots \otimes g_{p+1} + (-1)^{P+1} g_{p+1} \otimes g_0 \otimes g_1 \otimes \dots \otimes g_p$$

$$\Rightarrow \text{Hochschild chain complex } \{ \rightarrow A \otimes \bar{A} \rightarrow A \otimes \bar{A} \xrightarrow{b} A \} = C(A, A)$$

(not linear dual of Hochschild cochain complex)  $= C^*(A)$

- formula makes sense replacing  $g_0$  by an element of any

$$A\text{-bimodule } M \quad C_p(A, M) := M \otimes \bar{A}^{\otimes p}, \quad b: C_p \rightarrow C_{p+1}$$

$$H_p(A, M) = \text{Tor}_p^{A \otimes \bar{A}^{\otimes p}}(M, A) \quad \text{use bar resolution of } A \text{ as } A \otimes \bar{A}^{\otimes p} \text{ -not- } b.$$

$$C^p(A, M) = \text{Hom}_k(\bar{A}^{\otimes p}, M), \quad \delta: C^p \rightarrow C^{p+1}$$

computes  $\text{Ext}_{A \otimes \bar{A}^{\otimes p}}^p(A, M)$  Hochschild cochain complex.

Hochschild  $P$ -cocycle  $\bar{\tau}$  is  $\bar{\tau} \in C^P(A, A^*)$   $d\bar{\tau} = 0$

$$C^*(A, A^*) = C^*(A, A)^*$$

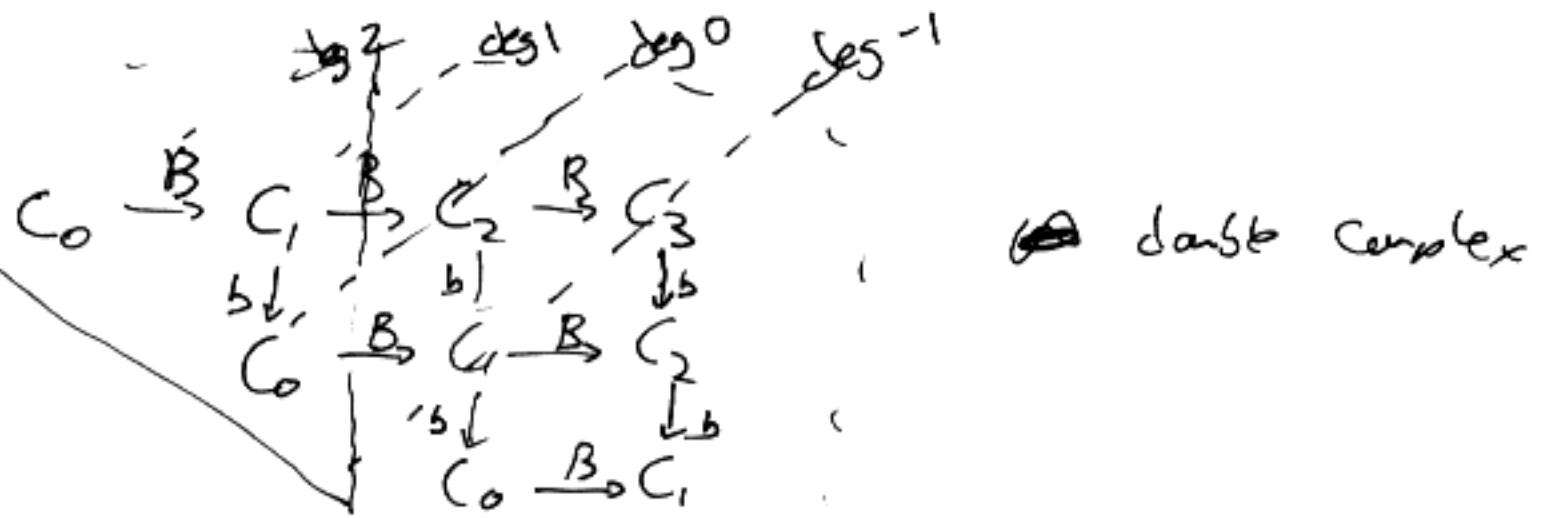
can also be considered as object of  $k$ -dual complex to  $C^*(A, A)$ .

$$\begin{array}{ccc} & \nearrow b & \\ A \otimes \bar{A}^{\otimes p+1} & & A \otimes \bar{A}^{\otimes p+1} \\ b \downarrow & \nearrow \delta & \downarrow b \\ A \otimes \bar{A}^{\otimes p} & & A \otimes \bar{A}^{\otimes p} \end{array}$$

$$\begin{aligned} & \mathcal{B}(g_0 \otimes g_1 \otimes \dots \otimes g_p) \\ & = \sum_{i=0}^P (-1)^{ip} g_0 \otimes g_1 \otimes \dots \otimes g_{i-1} \end{aligned}$$

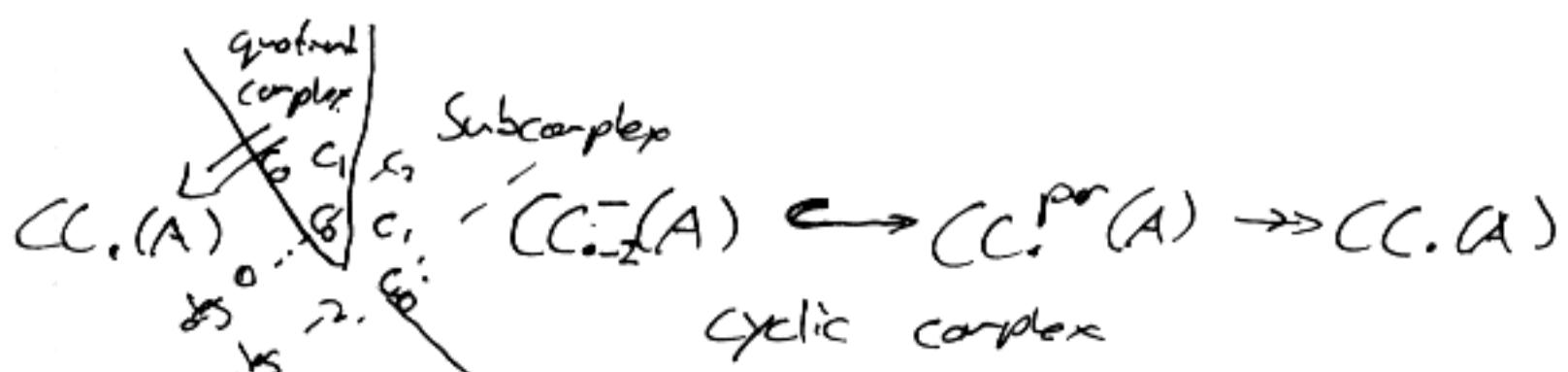
$$b^2 = \delta^2 = b\mathcal{B} + b\delta$$

$\Rightarrow$  double complex or  $C^*(A, A)$



Def  $CC_p^{\text{per}}(A) = \prod_{i \in p \pmod{2}} C_i(A)$  periodic complex

$b + B: CC_p^{\text{per}} \rightarrow CC_{p+1}^{\text{per}}$   $p \in \mathbb{Z}$



# B. Tsigan - NC Geometry II

1/30/03

$$C_n(A) = A \otimes \bar{A}^{\otimes n} \quad \bar{A} := A / k \cdot 1$$

$$C_n(A) \xrightarrow[B]{b} C_{n+1}(A)$$

$$b: a_0 \otimes \dots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i \otimes a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$$

b multiplies cyclic neighbors, B inserts 1 cyclically

$$B: a_0 \otimes \dots \otimes a_n \mapsto \sum_{i=0}^n (-1)^{ni} 1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

$$b^2 = BS + bB = B^2 = 0$$

$$A \otimes \bar{A} \quad a_0 \otimes a_1 \\ \downarrow \quad \downarrow \\ A \quad [a_0, a_1]$$

$$C_*(A) / b C_*(A) = \Omega'_{A/k} = \frac{c_* d_*}{c_* d_* + a_* b_*} \quad c_* d_* \text{ de Rham, } a_* b_* \text{ Kunneth}$$

$$\text{Getzler's notation: } (C_*(A)[u]), b + uB \quad |u| = -2, |b + uB| = -1 \\ = CC_*(A)$$

$$\begin{matrix} \downarrow & & & & \\ C_2 & \xrightarrow{uC_3} & \cdots & & \\ \downarrow & & & & \\ C_1 & \xrightarrow{uC_2} & \xrightarrow{u^2C_3} & & \\ \downarrow & & & & \\ C_0 & \xrightarrow{uC_1} & \xrightarrow{u^2C_2} & & \end{matrix}$$

$$CC_*^{\text{per}}(A) = CC_*(A)[u^{-1}]$$

$$CC_*(A) = \frac{CC_*(A)[u^{-1}]}{u CC_*(A)}$$

Why formal power series? want giving  
at degree zero part to extend to  
quasi of R/H object...

Given complex + endo of degree -1 (opposite to R/H)  
 $\Rightarrow$  construct the complex in way that sends gives to  
quasi

Ex. A commutative ring  $\Rightarrow$  do same to  $(\Omega^*, d, B=0)$   
 map  $\mu: a_0 \otimes \dots \otimes a_n \mapsto \sum a_0 da_1 \dots da_n$   
 gives morphism  $\mu \circ B = d_{DR} \circ \mu, \mu \circ b = 0$   
 $\mu: C_*(A) \rightarrow \Omega^*_{A/k}$

invariant under vector fields not anti-vector fields  $\Rightarrow$   
 correct get Riemann-Roch formula

$$CC_*(A) \rightarrow \Omega^*_{A/k}[B^{-1}], u \in DR$$

$$\begin{aligned} H C_n(A) &\rightarrow \mathbb{Z}_{A/k} \oplus \prod_{i \geq 0} H_{DR}^{n+2i}(A/k) \\ H C_n(A) &\rightarrow S^n_{A/k} / B^n_{A/k} \oplus \bigoplus_{i \geq 0} H_{DR}^{n+2i}(A/k) \\ H C_n^{\text{per}}(A) &\rightarrow \prod_{i \in \mathbb{Z}} H_{DR}^{n+2i}(A/k) \end{aligned}$$

Theorem (Loday-Gillan) If  $A = k[X]$ ,  $X$  regular & affine, then these are isomorphisms  
(and also  $H_n(A) \rightarrow \Omega^n_{A/k}$  Hochschild complex)

$$A = C^\infty(M) \Rightarrow \text{still true if understand } \otimes \text{ properly}$$

es  $C^\infty(M)^{\otimes n} := \begin{cases} C^\infty(M^n) \\ \text{or germs or jets at } \Delta \text{ of } C^\infty(M^n) \end{cases}$

Singular commutative case : take  $R \rightarrow A$

dg commutative free algebra resolution of  $A$

Then same picture holds : take  $n$ -forms  $\Omega^n_{R/k}$   
they form a dg  $\Omega$ .

e.g.  $\Omega^n_{R/k}$  is basically the cotangent complex  
(more precisely  $\Omega^n_{R/k} \otimes R$  is the target complex)  
— everything carries over for  $R$  instead of  $X$  —  
 $C_*(R)$  both quisas.

$$C_* \xrightarrow{(\#)} \Omega^*(R)$$

$$\begin{aligned} \text{Theorem} \quad \text{Over } k = \mathbb{C} \text{ & for any } X \text{ affine,} \\ H C_n^{\text{per}}(k[X]) \xrightarrow{\sim} \prod_{i \in \mathbb{Z}} H_{\text{sing}}^{n+2i}(X, \mathbb{C}) \end{aligned}$$

Draft Hochschild complex for  $A$  commutative :

$$C_*(A) = \text{Sym}_A^{\bullet}(\Omega^1_{A/k})$$

$\Omega = \Omega^1_A$  in smooth case,  
just in degree one

$\Omega = \text{cotangent complex} - \text{Kähler differentials of}$   
any resolution of  $A$ , then made into  $A$ -module.

There is also a curved construction of cotangent complex  
in Hochschild flavor.

$$\text{What is } \Omega : A \otimes A^{\otimes 2} (a, b) = a \otimes b \quad (a, b) = (a, c) + (a, c)$$

Universal relating algebras :

$$A \otimes F^3(A) \rightarrow A \otimes \Omega^2 A \rightarrow A \otimes$$

$F_3$  gave universal polynomial functors of degree three  
 - get Hochschild complex a lot more as its Synt

~~exact~~

So Hochschild complex has richer structure:

it's a commutative algebra :  $A \xrightarrow[A \otimes A]{\wedge} A \otimes A = \text{Hochschild}$

& in comm case  $A$  is a algebra over  $A \otimes A$ .

Dual  $A$ -module to  $A \Rightarrow$  associative (Yoneda) algebra  
 of Ext's, so Hochschild is a coalgebra

$\Rightarrow$  dg comm ~~Hochschild~~ algebra, ~~alg~~

$\rightsquigarrow$  dual to a universal envelope algebra.

this is origin of Synt ( $\Omega^1$ ) description

$\hookrightarrow$  Lie coalgebra

NC setting differential forms are not initial objects

but homology of something happens even in commutative  
 but singular case.

How to interpret singular cohomology: as hypercohomology  
 of de Rham complex

$$C(A) = \text{Synt}(\Omega^1) = A \otimes (\bar{T}A)^{\vee} \xrightarrow{\text{U.EA. of free Lie algebra}} \\ \text{CoUnivEA}(\underbrace{\text{Free}(\text{Lie}(A^{\vee}))}_{\Omega^1})$$

$$\text{Isom } C_n(MW(A)) \xrightarrow{+r} C_n(A)$$

$$a_0 \otimes \dots \otimes a_n \mapsto \sum (a_0)_{0,1} \otimes (a_1)_{1,2} \otimes \dots \otimes (a_n)_{n,0}$$

is a morphism of complexes (commutes with  $\delta, \beta$ )

& is a quism.

$e^2 = e$  idempotent in  $A$ , consider  $e \in C_0(A)$

$e \bmod bG$  gives class in  $A/[eAA]$  invariant of  
 projective module  $Ae$

$$C_1(A)$$

$$\downarrow b \in \mathbb{Z}$$

$$C_0(A)$$

$$C_2(A) \ni ch_1(e)$$

$$\downarrow b$$

$$C_1(A) \ni 1/e$$

$$\beta(e) = 0 \text{ and } \text{im}(g)$$

can lift to  $ch_1(e) \in G(A)$

$$(ze^{-1}) \otimes e \otimes e$$

Define  $ch_n(e) \in C_n(A)$  by

$$ch_n(e) = (-1)^n \frac{(2n)!}{n!} (e - \frac{1}{2}) \otimes \underbrace{e \otimes \dots \otimes e}_{2n}$$

$$ch(e) := \sum_0^\infty u^n ch_n(e) \implies \underline{(b+uB)ch(e) = 0}$$

$\Rightarrow ch(e)$  cycle in  $CC^-(A)$

Theorem (Karoubi-Connes)  $ch$  defines a morphism  $K_0(A) \rightarrow HC_0^-(A)$

$$\text{eg } A = C^\infty(M) \quad e^2 = e \in M_W(C^\infty(M))$$

$$\mu: CC_*(A) \rightarrow \mathbb{R}[[t^{-1}]], \text{ under}$$

$(N, e)$  vector bundle  $\implies$  canonical d.-formal form:  
 $e$  do e gives a connection on  $(N, e)$ ,  
~~e~~ get Chern character form of the connection  
 - projection of connection on ~~the~~ trivial bundle to d.-form  
 Summarise, may pick up curvature!

Conjecture that  $HC_0^-(\text{dg category of perfect complexes}) = H^1(X, CC_0^-(G))$  <sup>(B.Keller)</sup>

$$HC_0^-(X) = \prod_i H^1(X, \Omega_X^{i-1} \rightarrow \Omega_X^{i-1} \wedge \dots) \quad H^1(X, \Omega_X^{\wedge i} \mathbb{Q}[[t^{-1}]])$$

- set characteristic classes in here

$$\text{Hochschild: } \rightarrow \bigoplus_{i,j \in Ob(C)} (H_n(i,j) \otimes H_n(j,i)) \xrightarrow{b} \bigoplus_{i \in Ob(C)} End(i)$$

$$\text{Given } i \in Ob(C), ch(i) = I_i \in End(i)$$

$$i, j \text{ isomorphic } i \xrightarrow{g} j \quad g^{-1} \otimes g \xrightarrow{b} I_j - I_i$$

$\Rightarrow$  well defined up to isomorphism

Topics 1. For  $g \in GL_N(A)$   $tr g^{-1} dg \in \bar{R}_{A, \mathbb{R}} \implies ch(g) \in HC_0^-(A)$   
 $K_0(A) \rightarrow HC_0^-(A)$   
 & higher versions  $K_n(A) \rightarrow HC_n^-(A)$

2. Comes notation for  $HC_*$ :  $F_+: H_+ \xrightarrow{\sim} H_-$  is functor  
 on operator, and on algebra  $A$  acts on  $H_+, K_-$ .  
 $e^2 = e \otimes A$  projector  $F_+$  not  $A$ -morph.

Simple case  $F_+$  invertible so  $\text{Ind}_{F_+} = 0$ , but can form  $eF_+e : eH_+ \rightarrow eH_-$ , can have index ...  $\text{Index}(eF_+e) = \langle (\text{cycle}_F, ch(e)) \rangle$

assume  $F_+, e, eF_+$  fin dimensional operator (ie  $e, F$  commute mod fin dim operator)

$$3. H_0(\text{cyl}(A)) = \text{Sym}(H(A), A)$$

$\mathfrak{g} \in GL_n(A)$  --- for simplicity  $\underline{\mathfrak{g}} \in GL_n(A)$

$$\underline{g}'dg \in \mathbb{R}_A^{\times} \leftrightarrow g' \otimes g \in A \otimes \bar{A}$$

$$ch_1 = g' \otimes g \xrightarrow{B} 1 \otimes g' \otimes g - 1 \otimes g \otimes g^{-1}, \quad \begin{matrix} b \\ \text{look for satisfying } b \\ \text{annihilating } h_1 \end{matrix}$$

$$\text{Claim: } B(g' \otimes g) = b(g' \otimes g \otimes g' \otimes g)$$

$$- \text{ keep same } B(\cdot) = b(\cdot)$$

$$(ch_n(g)) = \ln g^{-1} \otimes g \dots \underset{n \text{ terms}}{\underbrace{\otimes g^{-1} \otimes g}} \quad \alpha_n \in \mathbb{Q} \text{ some coeff}$$

Claim:  $ch(g) = \sum_{n=0}^{\infty} ch_n g^n$  gives cycle in  $CC_-(A)$

$$(b + uB) ch(g) = 0.$$

Theorem (Connes-Karoubi):  $ch$  induces  $K_1(A) \rightarrow H_1(A)$

$$(K_1(A) = GL_\infty(A) / [GL_\infty(A), GL_\infty(A)])$$

tr  $d \log(g)$  is additive!

$$\text{tr}(d \log) = d \log(\det) \text{ for matrices over commutative ring } H,$$

$$g' \otimes g + h'' \otimes h - (gh)' \otimes gh \in C(A) \Rightarrow ch : K_1 \rightarrow H_1$$

All char classes form  $K$  theory factor through  $H_0(GL_\infty)$ , & can map this to  $H_1$ .

Graph complex: Let  $G = GL(A)$

$$C_*(G) \xrightarrow[\text{easy}]{} C_*(A) \quad \text{ Hochschild}$$

$$\xrightarrow[\text{harder}]{} CC_-(A) \quad \text{ negative cycle}$$

$$(g_1, \dots, g_r) \mapsto g_1^{-1} \otimes \dots \otimes g_r \otimes \dots \otimes g_r$$

B. Tsygan - Cyclic & Hochschild Homology, Chern Character

3/24/03

A assoc unit alg /  $k \rightarrow Q$  with counit

$$\bar{A} = A/k \cdot 1 \quad C_n(A) = A \otimes \bar{A}^{\otimes n}$$

$$b: C_*(A) \rightarrow C_{*-1}(A), \quad B: C_n(A) \rightarrow C_{n-1}(A)$$

cyclically multpl

$$A \otimes \bar{A}^{\otimes 2}$$

$\downarrow b$

$$A \xrightarrow{B} A \otimes \bar{A} \quad a_0 \otimes a,$$

$$a_0 \mapsto 1 \otimes a_0 \quad \begin{matrix} \downarrow b \\ A \end{matrix} \quad \begin{matrix} I \\ a_0 \otimes a, -a, a_0 \end{matrix} \quad ("[ , ]")$$

$$("d") \quad \quad \quad ("Leibniz identity")$$

cyclically insert unit

$$a_0 \otimes a_1 \otimes a_2$$

$\downarrow$

$$-a_0 \otimes a_1 a_2 + a_0 a_1 \otimes a_2 + a_2 a_0 \otimes a_1$$

(Drinfeld) Consider  $A$  as  $A \otimes A^\circ$ -module, and as  $(A \otimes A^\circ)^\circ = A \otimes A^\circ$

$$A \overset{L}{\otimes} A = (C_*(A), b) \quad \text{Hochschild complex} \quad \& \text{ "cochain"}$$

Additional structure: endomorphism of degree  $-1$

(vs opposite grad. fun  $\circ C^k = C_{-k}$ , to make  $b$  have degree 1)

$A \overset{L}{\otimes} A \in D^b(\text{Vect})$  = square of cohology groups, naturally,

but naturally (crystalline) carries more structure:

$$A \overset{L}{\otimes} A \in D(k[B]\text{-mod}) : \quad k[B] \text{ deg } B = -1, \quad B^2 = 0$$

supercomm algebra

Now can form more complexes out of these, via Koszul duality,

- cyclic, negative & periodic:

$$F_{cycl}, F_{negative}, F_{periodic} : D(k[B]\text{-mod}) \rightarrow D(\text{Vect})$$

$$F_{periodic} : D(k[B]\text{-mod}) \xrightarrow{\sim} D(k[u]\text{-mod}) \quad \begin{matrix} \deg u = 2 \\ \text{torsion} \wedge \\ \wedge \end{matrix} \quad \begin{matrix} \deg u = 2 \\ \text{cl}_u = 0 \end{matrix}$$

$$D_{torsion}(k[-]) :$$

action of  $u$  loc nilpotent on  $d'$  cohology grps.

$$D(k[u]\text{-mod})$$

$$(Tsygan) HC^- CC_-(A) = (C_*(A)[[u]], b + uB) \quad \deg u = -2$$

$$HC^{per} CC_*^{per}(A) = (C_*(A)[(u)], b + uB)$$

$$HC CC_-(A) = CC_*^{per}/CC_- \quad C_*(A) = CC_*/uCC_-$$

More economical version: let  $\bar{c}: A^{\otimes n+1} \rightarrow A^{\otimes n+1}$   
 $\bar{c}: a_0 \otimes \dots \otimes a_n \mapsto (-)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$

Fact: 1.  $\text{Im}(1 - \bar{c})$  is a b-subcomplex (though  $\bar{c}$  doesn't commute with b)

$$\text{a (so!) } C_n^\lambda := A^{\otimes n+1}/\text{im}(1 - \bar{c}) \text{ computes } H_0(A) \\ = H_0(C_*(A))$$

- more like algebra friendly definition, but not as close to Hochschild complex or forms.

A dg category (or  $A_\infty$  category)

$$C_n(A) = \bigoplus_{i_0, i_1, \dots, i_n \in \text{Ob } A} \text{hom}(i_0, i_1) \otimes \text{hom}(i_1, i_2) \otimes \dots \otimes \text{hom}(i_n, i_0)$$

→ generated by n-simplices in  $\text{Nerve of } A$

-- ie circles ( $\text{cycle}^n$ ) in the category

Formulas for  $b, B$  as above, replace products with coproducts. In dg setting replace  $b \rightsquigarrow b + b$

Hochschild complex comes from simplicial complex in the category

Cyclic complex comes from  $\zeta$

Given: Sheaf of rings  $A$  on space  $M$

$\Rightarrow A = \text{perfect complexes}$

(Drinfeld),  $X$  schae/field & quasi-compact & quasiseparated

"distributors" on  $D_{\text{perf}}(X)$  - complexes of sheaves with quasicoherent cohomology sheaves

"Schurz class" on  $D_{\text{perf}}^b(X)$  - most basic derived category, better than coherent one in singular settings.  
 function  
category

$X$  quasiprojective: a perfect complex: object of derived cat which can be represented by complex of vector bundles  
 More generally: full subcategory of  $D_{\text{perf}}$   
 s.t. on every open affine it is quasiisomorphic to a finite complex of vector bundles

No reasonable defns of  $C_*$ ,  $C_*$ , etc for triangulated cat

$\rightarrow$  need dg or Ab category.

$\exists$  def $\circ$  of dg category structures on  $D_{\text{coh}}^{\leq}, D_{\text{perf}}$ :

(can consider inside  $D(X\text{-mod}) = \text{Ho}(\text{Injective complexes})$ )

All  $X\text{-mod} \leftrightarrow$  homotopy category of injective complexes.

Injective complexes form a DG category, can look at full subcategory of perfect complexes

- injective complexes are pathological if  $X$  not Noetherian!

Theorem (B.Keller)  $CC_-(A_{\text{perf}}) \simeq H^*(X, \mathcal{C}_-(A))$

$X$  ringed space (at least such statement for  $C_-, CC_-$  for negative & periodic probably need finiteness conditions)

i.e. Hochschild cohomology of  $A_{\text{perf}}$  is  $H^*(X, \Omega_X \otimes^L \Omega_X) \dots$

$X$  smooth scheme,  $A = \mathcal{O}_X$

$$HC_-(A) \xrightarrow{\sim} H^*(X, \mathcal{R}_X^{\bullet}[\Omega]) \text{ under}$$

$$HC_{\text{per}} \xrightarrow{\sim} H^*(X, \mathcal{R}_X^{\bullet}[\Omega, \Omega]), \text{ under}$$

$$HH_* \xrightarrow{\sim} H^*(X, \mathcal{R}_X^{\bullet}), \text{ } \bullet \text{ differential}$$

$$(HH_* = \bigoplus_{j-i=k} H^i(X, \Omega^j))$$

Filtration by powers of  $\Omega \implies$  spectral sequence

$$E' = HH_*(\Omega) \implies HC_{\text{per}}$$

NC version of Hodge-deRham spectral sequence

$$\text{Hodge} \hookrightarrow \bigoplus H^i(X, \Omega^i) \quad \text{deRham} \hookrightarrow H^*(X, \Omega^{\bullet}_{\text{dR}})$$

$\mathbb{Z}/2$ -graded

So Hodge-deRham s.s. makes sense for any dg category!

$X$  smooth proper in char 0  $\Rightarrow$  ss degenerates

Kontsevich-Schechtman Conjecture:

$A$  - a stack of rings on  $X$ . Assumptions:

- $A = A_{\text{perf}}$
1. smoothness:  $A$  is perfect as  $A \otimes A^{\text{op}}$ -mod
  2. compactness:  $\dim HH^*(A) < \infty$

Conjecture The Hochschild  $\rightarrow$  cycl. s.s. degenerates at  $E'$

- (Defn!) NC geometry versus :
- $\mathcal{A}$  DG category is said to be proper if  $\forall X, Y \in \mathcal{A} \dim(\oplus_{i=1}^{\infty} \text{Hom}(X, Y))$  is finite &  $\mathcal{A}$  is finitely generated in Karabia sense.
  - $\mathcal{A}$  is said to be smooth if the diagonal bimodule  $\mathcal{A}_{\text{diag}}$  is a perfect  $\mathcal{A} \otimes \mathcal{A}^\circ$ -module ( $\Rightarrow$  fin. gen.)
- $\mathcal{A}_{\text{diag}}$ : if  $\mathcal{A}$  has one object this is an dgq as a sheaf. An  $\mathcal{A} \otimes \mathcal{A}^\circ$ -module is a bimod  $\mathcal{A} \times \mathcal{A}^\circ \rightarrow$  coproducts of  $\mathcal{A}$ -modules. Have a canonical such!  $(X, Y) \mapsto \text{Hom}(Y, X)$

Some  $X$  is smooth iff  $\mathcal{O}_X$  is a perfect complex!

For stable DG category assoc to  $X \times X$  is tensor product of that assoc to  $X$  with itself.

(Tsyg) Formal deformations :  $\mathcal{A}^t$  family of categories over  $\mathbb{C}[[t]]$

$$\text{hom}_{\mathcal{A}^t}(i, j) = \text{hom}_{\mathcal{A}}(i, j)[[t]]$$

Claim conjecture true for  $t \Rightarrow$  true for  $\mathcal{A}^t$ !  
 - dim of Hochschild homology can only drop,

$$\dim_{K[[t]]} HH_*(\mathcal{A}^t) \leq \dim_{\mathbb{C}[[t]]} HH_*(\mathcal{A})$$

But  $HC_*^{\text{per}}$  is RIGID : invariant under deformation  
 $HC_*^{\text{per}}(\mathcal{A}^t) \cong HC_*^{\text{per}}(\mathcal{A})[[t]]$ ,  
 --- cyclic homology is "topological" (Gauge-Main connection)  
 so  $HH_*$  has no choice but stay the same

Example  $A = C^\infty(M)([[t]])$ ,  $(M, \omega)$  symplectic. deformation quantized.  
 Before deformation:  $HH_*(A) = \Omega^0(M)$  not fin.

Theorem  $HH_*(\mathcal{A}^t) \cong H^{\dim M - i}(M, \mathbb{C}[[t]])$

fin. dim - just de Rham cohomology with vanishing ch. & S. S.  $HH \Rightarrow HC^{\text{per}}$  degenerates at  $E'$  for  $\mathcal{A}^t$ .

Lie algebra homology of Lie dg,  $C_*(\mathfrak{g}) = 1^{\circ} \text{ of}$

$$[x_1, \dots, x_n] = \sum_{i,j} (-1)^{i+j-1} [x_i, x_j] \wedge \dots \wedge \widehat{x_i} \wedge \dots \wedge x_n$$

homology is  $H_*(\mathfrak{g})$

Now take  $\mathfrak{g} = \text{gl}(A) = \varinjlim_N \text{gl}_N(A)$

Theorem  $H_*(\text{gl}(A)) = \text{Sym}^* (\text{HC}_{-1}(A))$

More specifically  $C_*(\text{gl}(A))_{\text{gl}(k)} \xrightarrow[\text{coinvariants}]{} \text{Sym}^* (\text{C}_{-1}^\lambda(A))$

(Passing to  $\text{gl}(k)$  coinvariants changes complex by a quism)

- follows from reductivity of  $\text{gl}(k)$  & reductive action

for small homotopy groups suffices to take  $\text{gl}_n(A)$  n big enough.  $\text{gl}(k)$  reductive

& acts trivially on its cohomology  $\rightarrow$  can take coinvariants.

Map:  $a_0 \otimes \dots \otimes a_n \mapsto E_{01}^{a_0} \wedge E_{12}^{a_1} \wedge \dots \wedge E_{n-1,n}^{a_{n-1}} \wedge E_{n0}^{a_n}$   
 $\text{C}_n^\lambda(A) \ni \text{mod in } (1-\epsilon)$

$E_{ij}^a = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$  matrix units

Product in  $\text{Sym} C_1$   $\leftrightarrow$  concatenation

$$\left( \begin{array}{c} (..) \\ (..) \end{array} \right)$$

Invariant theory gives isomorphism!

eg  $H_1(\text{gl}(A)) = \text{gl}/[\text{gl}, \text{gl}] \xrightarrow{\text{Tors}} A/[A, A] = \text{HC}_1$

$H_2(\text{gl}(A)) = A^2/[A, A] \oplus \text{HC}_2(A)$

$\text{C}_1^\lambda(A) = \text{Prim } C_*(\text{gl}(A))$  - due to Bloch...

Gelfand-Fuks's conjecture: case  $A = \mathbb{C}((t))$  ..

Why symmetric algebra?  $H_*(\mathfrak{g})$  always coalgebra  
 from  $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ . But we have almost " $\text{gl} \otimes \text{gl} \rightarrow \text{gl}$ "

( $\frac{d}{dt} \otimes \frac{d}{dt}$ )  $\text{gl}_S \otimes \text{gl}_S \rightarrow \text{gl}_{S \times S}$  is contractible set, identify  $S \times S \cong S$   
 defined up to conjugation  $\rightarrow$  morphism on level of  
 coinvariants already  $\Rightarrow$  Hopf's structure, coproduct commutes  
 $\rightarrow$  (Milnor-Moore) Hopf's structure, coproduct commutes

# B.Tsygan - Cyclic Homology & NC Geometry VI

3/6/03

Recall:  $I \in A$  assoc algebra /  $k$   $\oplus I \subset k \Rightarrow I$  (or dg category)

$$C_n A = A \otimes \bar{A}^{\otimes n} \xrightleftharpoons[b]{\cong} C_{n-1} A \quad n \geq 0 \quad b^2 = bB + BB = B^2 = 0$$

$$CC^{\text{per}}(A) = C(A)(w), \quad b + uB \quad CC^-(A) = C(A)[[w]] \quad b + uB$$

$$\begin{aligned} C_0 / b C_1 &= A / [A, A] && \text{linear span of commutators} \\ C_1 / b C_2 &= \Omega_A \end{aligned}$$

$$CC^{\text{per}} / C_{\geq 2}(A)(w)) + b \subset C_2(A)(w))$$

- "de Rham complex + feedback!"

0. For free algebra  $A$  projector onto the truncation is a quasi-isomorphism:

$$CC^-(A) \xrightarrow{\text{proj}} \left\{ \begin{array}{l} A \rightarrow \Omega_A \\ A \rightarrow \Omega_A' \end{array} \right\} \text{ is a quasi-isom.}$$

similarly for  $CC^{\text{per}}$ ,  $C$ . whole functor is "closed" from here.

1.  $A = k[V]$  affine nonsingular variety :  $CC$  is quasi-isomorphic to de Rham  $(CC^-(A), b+uB) \xrightarrow{\sim} (\Omega^*[[w]], u \partial_w)$

Marita invariance :  $A \xrightarrow{\text{Morita}} B \Rightarrow$  same Hecke & cyclic homologies  
J. Getzler  $HC^-(A) \xrightarrow{\sim} HC^-(B)$  (perfect complexes of  $A$ -modules  
(B.Keller))

• Chern Character :  $ch: K_0(A) \rightarrow HC_0(A)$ ,  $K_n(A) \rightarrow HC_n(A)$

• Relation to Lie algebra homology I :

$$H_*(gl(A)) \cong S^*(HC_{-1}(A))$$

$A \supset I$  nilpotent two-sided ideal  $\Rightarrow$  relative K-theory  $K_i(A, I)$   
 $K_{i+1}(A/I) \xrightarrow{\sim} K_i(A, I) \rightarrow K_i(A) \rightarrow K_i(A/I) \rightarrow K_{i+1}(A/I)$

$$HC_i(A/I) \rightarrow HC_{i-1}(A, I) \rightarrow HC_{i-1}(A) \rightarrow HC_{i-1}(A/I) \rightarrow HC_{i-2}(A, I)$$

Goodwillie :  $HC_{i-1}(A, I) \xrightarrow{\sim} K_i(A, I) \quad i > 0$

so while K-groups are hard the difference between them is governed by cyclic homology

$$\begin{array}{ccccc} \mathcal{C}_i & \hookrightarrow & \mathcal{C}_i^{\text{per}} & \rightarrow & \mathcal{C}_{i-2} \\ \mathcal{H}_i(A, I) & \xrightarrow{\quad \text{Fr. } \dots \quad} & \mathcal{H}_i^-(A, I) & \xrightarrow{\quad \text{Fr. } \dots \quad} & \mathcal{H}_i^{\text{per}}(A, I) \\ & & \uparrow \text{ch} & & \parallel \\ & & K_i(A, I) & & \end{array}$$

$$so \quad K_i(A, I) \xrightarrow[\text{ch}]{} \mathcal{H}_i^-(A, I) \simeq \mathcal{H}_{i-1}(A, I)$$

$$A = \mathbb{Q} + I \text{ nilpotent ideal} \rightarrow H_*(GL(I)) \xrightarrow{\sim} H_*(\text{ggl}(I))$$

$\text{ggl}(I) = \ker(GL(\mathbb{Q} + I) \rightarrow GL(\mathbb{Q}))$        $K\text{-theory}$        $\text{cyclic homology}$

$A$   $\mathbb{Q}$ -algebra

$$H_*(GL(A), \mathbb{Q}) \simeq S^*(K_*(A)_\mathbb{Q})$$

$$H_*(\text{ggl}(A), \mathbb{Q}) \simeq S^*(\mathcal{H}_{*-1}(A)) \quad - \text{ trying to describe as}$$

"Tangent space to  $K$ -theory"...

Another way to describe tangent space to  $K$ -theory would be  
 $\ker \left( K_*(A[\varepsilon]) \xrightarrow[\varepsilon=0]{} K_*(A) \right) = K_*(A[\varepsilon] / \varepsilon A[\varepsilon])$

Hochschild homology  $HH_{*-1}(A) \cong \ker(H\mathcal{C}_i(A[\varepsilon]) \xrightarrow{IS} H\mathcal{C}_{*-1}(A))$        $\text{Goochlike}$

- i.e. one says either one is "tangent to  $K$ -theory,"

- Hochschild of tensor product is tensor of Hochschild, so

$$HH(A[\varepsilon]) = HH(A) \otimes HH(k[\varepsilon])$$

(compute  $HH_k(k[\varepsilon]) = k$       Now use  $HH \Rightarrow HC$   
 spectral sequence:  $B$  is also multigraded  $HH_* \rightarrow HH_{*-1}$   
 (a derivation)

### Relation to Lie Algebra Homology II:

Assume  $\mathfrak{g}$  acting on  $A$  by derivations

$$\text{let invariant trace } \overline{t}: A / [A, A] + \mathfrak{g}A \longrightarrow k$$

$\Rightarrow$  produce cocycles in end kit now

$$C_*(\mathfrak{g})[[u]] \xrightarrow{\quad \text{u is as differential} \quad} \text{Hom}_{k[[u]]}(CC_*(A)_*, k[[u]])$$

$$C_*(\mathfrak{g}) = \Lambda^\bullet \mathfrak{g} \text{ Chevalley complex}$$

Commutative case:  $A = C^*(M)$   $\text{sg} \rightarrow \text{Vect } M$   
 $C(A) = \int_M a \cdot \text{vol}$  where vol is a sg-intr volue form

$$C_*(A) = (\Omega_{\text{in}}(M), d_{\text{DR}})$$

$C_n(\text{sg}) \Rightarrow D_1, \dots, D_n \rightarrow \text{product } \Omega_{\text{in}}^{[n]} \rightarrow \mathbb{C}[u]$   
 $w \mapsto \int_{D_n} z_0 \cdots z_n(w) \cdot \text{vol}$   
 integrate contractions with vector fields

### NC Calculus:

Commutative case:  $\text{sg} \subset \text{Vect } M \Rightarrow \text{sg} \text{ acts on } \Omega^* M \text{ etc}$   
 - in fact a much richer (dg) Lie algebra acts

NC case:  $\text{sg} \subset \text{Der}^A$  acts on  $C_* A$  etc. What is the richer structure present here?  $\rightarrow$  NC calculus

Example: construction of cyclic cocycles from index theory of Fredholm operators

e.g.  $M$  manifold

$\Lambda^* = \Lambda^* T_m$   
 polyvectors  
 $\Omega^* = \Omega_{\text{in}}^* = \text{forms}$

$\Lambda^*$  comm.  
 wedge &

Schouten bracket

$\Lambda^{\otimes p} \xrightarrow{\text{Pf}} \Lambda^p$  action  
 $\Lambda^* \otimes \Lambda^* \rightarrow \Lambda^* \text{ Lie bracket}$   
 $a \otimes \omega \mapsto 2_a \omega$  multiplicative  
 $a \otimes \omega \mapsto L_a \omega$  Lie algebra

a Calculus: • two graded spaces  $\Lambda^*$  and  $\Omega^*$

•  $\Lambda^*$  a graded commutative algebra

•  $\Lambda^{*+1}$  a graded Lie algebra

setting:  $[a, b]_c = [a, b] + (-1)^{|a|(|b|-1)/2} b[a, c]$

$\Lambda^*$  Gerstenhaber  
 algebra

•  $\Omega^*$  is a graded  $\Lambda^*$ -module (in comm sense),  $2_a w$

•  $\Omega^*$  is a graded Lie algebra  $\Lambda^{*+1}$ -module,  $L_a w$

satisfy  
 $[\Omega_a, \Omega_b] = [\Omega_a, \Omega_b], L_{ab} = L_a \Omega_b + (-1)^{|a|} \Omega_a L_b$

### Precalculus

•  $d: \Omega^* \rightarrow \Omega^{*+1}, d^2 = 0, [d, \Omega_a] = L_a$  (calculus formula)  
 $\Rightarrow$  calculus if also have such  $d$

Precalculus  $\leftrightarrow$  Hochschild Calculus  $\leftrightarrow$  Cyclic  
 Forget about multiplication of forms: no NC analog...

Example  $A$  any ring  $\Rightarrow$  naive calculus  
 $\text{Calc}_0^*(A) : \Omega_{(0)}^* A = HH_0(A)$

$\Lambda_{(0)}^*(A) = HH^*(A)$  ...  $HH^{*+1}$  is a graded Lie algebra  
with Gerstenhaber bracket.  $\delta = B$

$HH$  Cup Product  $\Rightarrow$  product on  $\Lambda^*$   
Cup product gives 2

DG calculus: calculus  $\Lambda^*, \Omega^*$  with extra differential  
 $\Lambda^* \xrightarrow{\delta} \Lambda^{*+1}, \Omega^* \xrightarrow{\delta=b} \Omega^{*-1}$  respecting all structures.

Conjecture, namely! get dg calculus  $\Lambda^* = C(A, A)$ ,  
 $\Omega^* = C_*(A, A)$  with  $\delta = b$

Try ~~Prop~~: L, J:  $C^{p+q}(A, A) \otimes C^{q+1} \rightarrow C^{p+q+1}$  is indeed  
a dg Lie algebra, &  $(C_*, b)$  is a dg module  
(completely explicitly) so Lie structures work,  
 $B: C_* \rightarrow C_{*+1}$  a module map.

So just need multiplication on  $C^*$  & action of this on  $C_*$ , for  
Multiplication  $\nu$  exists & is associative but  
not commutative (only on cohomology level)  
Leibniz identity fails.

• General remark on calculus: What Lie algebra structure  
arises from a calculus?  
( $\Omega^* = \Lambda^{*+1}, \delta$ ) graded Lie alg,  $(\Omega^*, b)$  graded module  
Introduce  $\Omega^*[[\eta]]$   $|\eta| = 1$  &  $\eta^2 = 0$   
with differential  $\delta + \frac{\partial}{\partial \eta}$  new dg Lie alg  
or  $(\Omega^*[[\eta]][[u]], \delta + u \frac{\partial}{\partial \eta})$  - acts on  $(\Omega^*[[u]], b + u\delta)$   
 $x + \gamma \eta \mapsto Lx + \gamma y$  - ie retain everything  
but multiplication of polyvector fields - all this  
happens on chain level... well almost.

Theorem There is a construction  $A \rightsquigarrow (\Lambda^* A, \Omega^* A, \delta, b)$   
algebra  $\rightsquigarrow$  dg calculus; before introducing  $\delta, b$  just standard  
construction in terms of only vectorspace  $A$  (not an algebra);

differentials  $b, d$  are defined in terms of multiplication in  $A$   
+ some numbers - which can be chosen if one  
has an associator: elt of homogeneous space for  
a complicated explicit pronipotent group (Grothendieck-Tichmüller)  
ie associators  $\rightsquigarrow$  calculi.

(formality: dg algebra [certain] comm alg is quiver to assoc. relations?

1. To every assoc algebra  $\rightarrow$  calculus algebra over  
a covariant 2-colored dg operad
2. This dg operad is quiver to one given above (Grothendieck)  
- quiver depends on choice of associator.

1. The  $\Lambda$  part  $\rightarrow$  (deformations of  $C^\infty(M)$   $\hookrightarrow$  formal Atiyah structure)  
Formality theorem  $\pi \in \mathbb{R}^{\gamma_m}$   
 $[\pi, \pi] = 0$

So consider  $A^\pi = C^\infty(M)[[t]]$ ,  $*_\pi$  quantization

Can consider  $CC_*(A^\pi) \xrightarrow{\sim} \Omega^m[[t, u]]$  under  $+ t \cdot L_\pi$   
 ~~$\mathcal{C}(A^\pi)$~~   $t \mapsto \sqrt{\pi} t$

So describe traces on deformed alg  
(Grothendieck / Gauss-Manin etc case just an act of  $g[[\gamma]][[u]]$ )

get generalization of Atiyah-Singer...

Connes-Moscovici: renormalizing op acts on choices just  
for  $(g[[\gamma]][[u]], f + u g)$  structure.

Strong formal resemblance to BV formalism,  $u = h$ .

Cyclic objects of a category ( $\text{Cyc}$ ): simplicial object  
+ additional structure. Cyclic space  $X \rightarrow |X|$  has  
circle action

Generator of circle is related to  $B\text{Eff}'(S')$ ,  $u \in H_{S'}^2(\ast)$

Free loop space  $LY \Rightarrow H^0(LY)$  is Harlisch's  
homology of something  $\cong$  classifying

## B. Toen Cyclic objects

3/3/03

$$\begin{array}{ccc}
 \text{cyclic} & \xrightarrow{+} & \text{cyclic} \\
 \text{cyclic} & \xleftarrow{\text{d}_n} & \text{cyclic} \\
 \text{cyclic} & \xrightarrow{\text{d}_n} & \text{cyclic} \\
 \text{cyclic} & \xrightarrow{\text{d}_n} & \text{cyclic}
 \end{array}$$

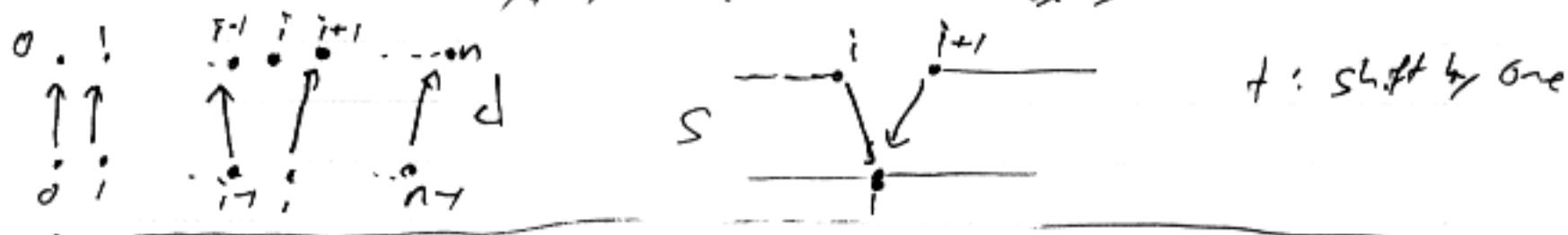
$\xrightarrow{i \geq 0}$

Not do it degenerate, start

$$\begin{array}{ccc}
 \text{d}_i : A_n^{\#} \rightarrow A_{n-i}^{\#} & \text{design} & s_i : A_n^{\#} \rightarrow A_{n-i}^{\#}, \quad 0 \leq i \leq n \\
 t : A_n^{\#} \xrightarrow{\sim} A_n^{\#} & t^{n+1} = 1 & d_n = d_0 \circ t
 \end{array}$$

$A^{\#}$  = cyclic nerve of a category : built of all cyclic objects  
 $[n] \xrightarrow{\text{fun}(i_0, 1) \times \dots \times \text{fun}(i_n, i_0)}$

Geometrically :  $X : \Delta^0 \rightarrow \text{sets simplicial}$        $Y : \Lambda^0 \rightarrow \text{sets cyclic}$   
 $\gamma_n = X([n]) = \text{Funct}([n], \mathcal{C})$   
 $Y([n]_{\text{cyd}}) = \text{Funct}([n]_{\text{cyd}}, \mathcal{C})$



Another construction of  $S'$  acts on geometric realization of cyclic nerve  $|C^{\#}|$  :

Category  $S'$  : objects = points of  $S'$ , morphisms : homotopy classes  
of ~~continuous~~ continuous paths  $x_0 \rightarrow x_1$ ,  $\dots$ ,  $x_n$

$|C^{\#}|$  = Piecewise constant functors  $S' \rightarrow \mathcal{C}$

Question: or paths all go to one object, functors go to identity,  
- really think of as functor defined outside  
some (not fixed) finite set, constant on copoints.

Dinfeld  $C$  category  $\rightarrow$  nerve  $NC \rightarrow INC /$  factor each  
What is a point of  $(NC)$  ?

Draw diagram  $\xrightarrow{x_0} \xrightarrow{x_1} \xrightarrow{x_2} \dots \rightarrow$

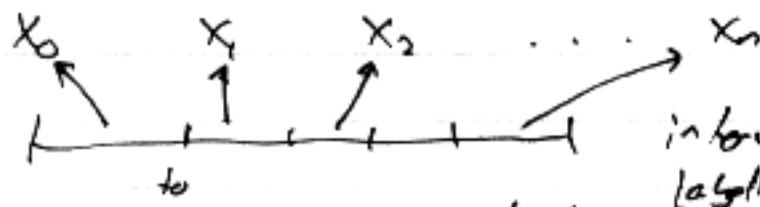
$$I = [0, 1] \quad \text{Simplex } \Delta^n := \{(t_0, \dots, t_n) \in I^n \mid t_0 + \dots + t_n = 1\}$$

better coordinates for theory of simplicial sets

Simplices have ordered vertices — realize by ordered sets of reals!

$\leftrightarrow$  point of  $\Delta^n$  is a partition of a segment  $I = [t_0, t_1, \dots, t_n]$

Part of  $|NC|$ : morphisms with lengths



interval with partition, each segment labelled by an object

$|NC|$ : piecewise constant functions from segment to  $C$ .

Symmetry  $S^p$  = order preserving homeomorphisms of  $I$

Cyclic case: replace segment by circle, get obvious  $S'$  action.

Symmetry of oriented homeomorphisms of  $S'$ .

Tsyan



$$\{a_0, a_1, a_2 \text{ at } (t_0, t_1, t_2) \in \Delta^2\} \subset A_2^A \times \Delta^2 \subset |A|^3$$

Action of rotation:

$$\text{hom}(i_0, i_1) \times \text{hom}(i_1, i_2) (a_0, a_1) \text{ at } (t_0, t_1) \in \Delta^1 \rightarrow t_0$$

$$(1, a_0, a_1) \text{ at } (t, t_0, 1-t_0-t)$$



$$(a_1, a_0) \text{ at } (1-t_0, t_0) \in \Delta^1$$



$$(1, a_1, a_0) \text{ at } (t, 1-t_0, -) \quad \begin{matrix} 2\pi \\ \downarrow \\ 0 \end{matrix}$$

So at certain discrete points in time we perform cyclic rotation of elts in our tuple.

Linear case:  $A^A$  is cyclic vector space, consider as simplicial vector space  $\Rightarrow$  handbag

$$HH_0(A) = H_0(A^A). \quad \text{Circle acts!}$$

$$H_*(S') \times HH_0(A) \rightarrow HH_0(A)$$

Eg  $A =$  linear envelope of a nonlinear category  $A = k[C]$   
 Then  $A^{\wedge} = k[C^{\wedge}]$

Since geometric realization of  $C^{\wedge}$  have homology =  
 homology of linearization  $A^{\wedge}$ , get circle action

Generator of  $H_1(S')$  acts as B character  
 on  $HH_0(A)$ .

Given  $A_{\Delta^{\text{op-mod}}}$   $k^{\wedge}$   $\Delta$ -mod - with trivial category structure,  
 ie  $(k^{\wedge})^n = k$   
 $\lim_{\Delta^{\text{op}}} A = A \otimes k := \bigoplus_{n \geq 0} A_n / \langle \text{cat-a: faces} \rangle$  tensor of at & left  
 $\Delta$ -modules

Tensor over  $\Delta$ : Simplicial  $\otimes$  cosimplicial is required  
 $k^{\wedge}$  cosimplicial wrt identity

Usual tensor over  $\Delta$  gives bar construction:

$$\text{Tor}_n^{\Delta}(A, b) = n^{\text{th}}$$
 left derived functor

On the other hand  $d^{\text{cell}}$  complex  $d = \sum (-1)^i d_i : A_n \rightarrow A_{n-1}$

Lemma:  $H_n(A) \xrightarrow{\text{to}} \text{Tor}_n^{\Delta}(A, k)$

$A \otimes k = \bigoplus_{n \geq 0} A_n / \text{non-homologous relators} = \text{Tor}_0^{\Delta}(A, k) = H_0(A)$ .  
 - can take face maps till get to vertex,

so only see  $H_0$ .

Proof:  $R_n = \text{hom}_{\Delta}([n], ?) \in \Delta\text{-rel} \dots \Rightarrow$  resolution of  $k$ , in  
 $\Delta\text{-mod}$

$A$  a  $1^{\circ}$ -space = cyclic vector space

$\Rightarrow$  look at  $A \otimes k$ ,  $\text{Tor}_1^{\Delta}(A, k)$

Connes:  $H_0(A) = \text{Tor}_1^{\Delta}(A^{\wedge}, k)$

$k$ -linear category  $\xrightarrow{\text{forget}} \text{Cyclic spaces}$

$A \xrightarrow{\text{forget}} \text{Complexes}$

define cyclic homology:  $b = \sum (-1)^i d_i$

$B(\otimes \otimes \dots \otimes) = \sum \pm 1 \otimes \dots \otimes \otimes \dots : \text{dilute from nothing in } A$

$A_1$   
 $A_2$   
 $A_3$

$\bullet$   
 $\bullet$   
 $\bullet$

Proof of Connes' Theorem: Construct resolution of the  $\Lambda$ -module  $k$

[  $C\mathbb{P}^n$ : Tor in relative direction, Ext =  $\Delta$  direction ]

Relation to loops  $X$  topological space,  $X^S = \text{Map}(S, X)$

J. Jones

$X \xrightarrow{\Delta} X \times X \Rightarrow$  Construct a cocycle topological space with  $\Delta$  as comultiplication;  $X \xrightarrow{d_1} X^2 \xrightarrow{d_2} X^3 \dots$  cochain!

$d_0, d_1: x_0 \mapsto (x_0, x_0)$  degeneracy

cocomultiplication:  $a_0 a_1 \xrightarrow{c_{001}} a_0 a_0$

Degeneracy  $s_0: X^2 \xrightarrow{s_0} X^1$

$(x_0, x_1) \mapsto x_0$

$s_0: A \xrightarrow{s_0} A \otimes A$

$a_0 \mapsto a_0 \otimes 1$ )

omission of arrows

e.g.  $X^3 \xrightarrow{d_2} X^4$   $x_0 x_1 x_2 x_3 \mapsto x_0 x_1 x_2 x_3 x_3$

$X^S = \text{Map}([n], X)$  get cyclic space, but  $\Lambda$  self-dual,  
so cyclic & cocyclic are the same.

$\rightarrow$  Cocyclic topological space  $\tilde{X}$

Apply dual to geometric realization in homotopy colimit

$\Delta^{op}$ -space  $Y \rightsquigarrow \|Y\| = Y \times \Delta^\bullet$

cospinifical top space  $\mapsto$  topological space

$\Delta$ -space  $Z \rightsquigarrow \|Z\| = \text{hocolim}(\Delta, Z)$

= homotopy colimit

instead of coincident points, connect by boundary

•  $\|\tilde{X}\| \rightsquigarrow X^S$  has oracles! rays from triangle are forced to lie on one of edges by degeneracy

Now: take coalgebra  $S_\bullet(X)$  of chains on  $X$  (signs)  
 $\Rightarrow$  Cyclic cohomology of the coalgebra  $S_\bullet(X)$

Relative  
signature

$H^*(S_\bullet(X)) \cong H_*(X^S)$  (Ch)

Note to coalgebra dual  $\Rightarrow$  to Hochschild homology of algebra

(assume  $\pi_1 X = 0 = \pi_0 X$ )

$H^*(S_\bullet(X)) \cong H_*(X^S)$  (J. Jones)

$$\text{Proof } S_*(X^S) = k \cdot \{ \Delta \rightarrow X^S \} \\ = \text{Hom}_\Delta(S_*\Delta, S_*X^S)$$

$$S_*(X^S) = S_*(\|X^S\|) = S_*(\text{Hom}(\Delta, X^S)) \xrightarrow{\text{using}} \text{Hom}(S_*\Delta, S_*X^S)$$

$$= \text{RHom}(k, (S_*X)^S) = H^0(S_*X)$$

↑ dual to comes theorem  $\square$

Chas-Sullivan  $X$  manifold,  $\Omega^* X$  dg algebra,

can consider Hochschild & cyclic chains

$$C_*(\Omega^* X) \in H^0(\Omega^*(X)) \cong H^0(X^S)$$

→ computes cohomology of  $X^S$ , dual to  $S_*$ .

Chas iterated integral:  $C_*(\Omega^*(X)) \rightarrow \Omega^*(X^S)$

Assume  $X$  compact oriented.  $\Omega^*(X) \cong \Omega^{n+*}(X)$  as brackets  
Poincaré duality  $\Omega^* \cong \Omega^n$ .

→ Dual since  $H^0(\Omega^*(X)) \cong \text{Ext}_{\Omega^*(X)}(\Omega, \Omega^*)$

$$H^S_* X^S \xleftarrow{\sim} \text{Ext}_{\Omega^*(X)}^S(\Omega, \Omega^*) \cong \text{Ext}_{\Omega^*(X)}^n(\Omega, \Omega^*)$$

So loop space homology is calculated  $\rightarrow H^0(\Omega)[n]$

by Hochschild cohomology of  $\Omega$  (see homology which appears in deformation theory).

But this is a Gerstenhaber algebra ..., ?  
From its cohomological origin.

Also dual to homology  $\rightarrow$  jet operator  $B^*$   
of degree 1 dual to  $B \Rightarrow BV$  algebra.

Theorem: Given  $A$  with nondegenerate trace,  $H^0(A)$  is a  $BV$  algebra, &  $C^*(A)$  is a  $BV_{\infty}$ -algebra

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$$\bullet BG^S = G^S$$