Matt Ando - Twisted Umkehr Maps

Note Title 3/29/2009

D-brane charges

[w/ Blumberg, Gepner, Sati]

$$\begin{tikzcd} V \text{ gauge bundle} \arrow[d] & \\ D \arrow[r, "j"] & X \end{tikzcd}$$

submanifold "brane"

Minasian-Moore: to calculate charges

$$\begin{tikzcd} K(D) \arrow[r, "j_!"] & K(X) \end{tikzcd}$$

$$V \longmapsto j_! V \mapsto \int_X \hat{A}(X)\, ch(j_! V)$$

$\rightsquigarrow$ think of $j_! V$ as being the charge of the brane, living in $K(X)$

$j_!$: umkehr map.

Standard hypothesis to have such: $Spin^c$-structure on normal bundle $\nu(j)$

$Spin^c$: w_2 is obstruction to spin, obstruction to $Spin^c$ is Bockstein of w_2

$$\begin{tikzcd} & B\,Spin^c \arrow[d] & & \\ X \arrow[ur, dotted] \arrow[r, "\nu"'] & BSO \arrow[r, "w_2"'] & K(\mathbb{Z}/2, 2) \arrow[r, "\beta"'] & K(\mathbb{Z}, 3) \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "V"] \arrow[d, "i"'] & BSO \arrow[d, "\beta w_2"] \\ X \arrow[r, dashed, "H"] & K(\mathbb{Z},3) \end{tikzcd}$$

Freed-Witten: can choose deg 3 class on X sometimes, whose restriction to D is $\beta w_2(V)$.

But $H^3(X,\mathbb{Z})$ give twists of K-theory $\Rightarrow$ can form $-H$-twisted K-theory of X, $K(X)_{-H}$

$$\begin{tikzcd} \tilde{K}(D^{\nu}) \arrow[rr, "\substack{\text{twisted} \\ \text{Pontrjagin-Thom} \\ \text{map}}"] & & K(X)_{-H} \\ K(D) \arrow[u, "\substack{\text{Thom space} \\ \text{of normal bundle}}", "\wr"'] \arrow[urr, "j_!"'] & & \end{tikzcd}$$

So get a charge $j_! V \in K(X)_{-H}$

Constructions of twisted umkehr maps:

- In C^* algebra approach to K-theory goes back to Kasparov, Rosenberg, ...
- Using Atiyah-Segal on twisted K-theory - cf. Carey-Wang

Both depend on explicit model for K-theory & working deep in it.

We'd like general approach, to apply to other cohomology theories.

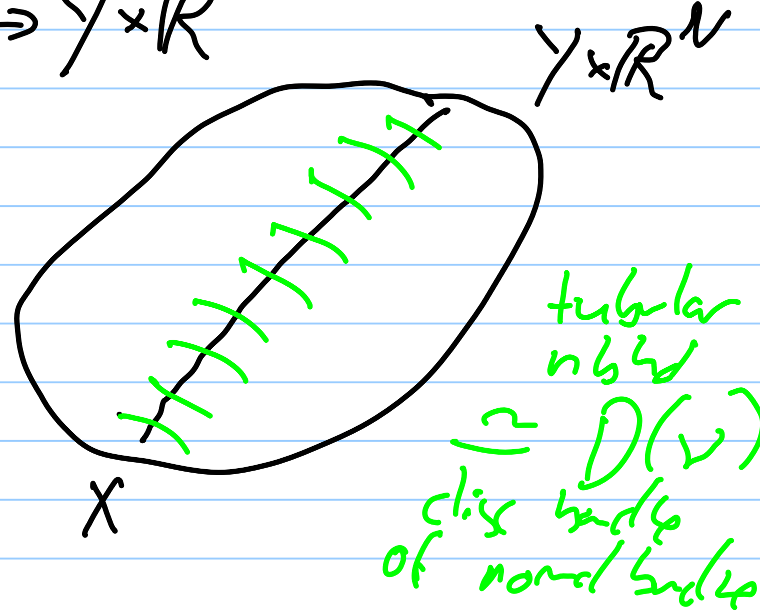
[elliptic cohomology / twist with H^4 class get M-brane charges..]

$X \xrightarrow{f} Y$ fiber bundle of compact manifolds

Convert into an embedding: choose $X \hookrightarrow \mathbb{R}^N$

$\rightsquigarrow \quad X \xhookrightarrow{\tilde{f}} Y \times \mathbb{R}^N$

Pontryagin-Thom



Now crush complement of normal bundle to a point

$\Rightarrow$ P-T collapse

$$(Y \times \mathbb{R}^N)^+ \xrightarrow[PT]{} D(\nu)/S(\nu) =: X^\nu$$

Thom space

So get functoriality on reduced cohomology theories:

$$\begin{array}{ccc} E(X^\nu) & \longrightarrow & E^\bullet((Y\times\mathbb{R}^N)^+) \\ \text{need Thom isom. for } \nu \;\; \| \wr & & \| \wr \quad \text{suspension} \\ E^*(X) & \xrightarrow[f_!]{} & E^{*-d}(Y) \quad \text{with degree shift} \end{array}$$

$d = \dim$ of fiber

E	$Y = pt$	general Y	Thom iso
H		$\int_{\text{fiber}}$	Tf oriented (tangent to fiber)
K	index of elliptic operator	families index theorem	Tf — Spin (KO); Spin^c (K)
Ell	elliptic genus	families elliptic genus	Tf string structure ($w_2 = 0$, $\frac{p_1}{2} = 0$) Ando-Hopkins-Rezk

Problem: no obvious "families" structure to this story - important for twisting

$$\begin{array}{ccc} (Y\times\mathbb{R}^N)^+ & \longrightarrow & D(\nu)/S(\nu) \\ \big\downarrow & & \\ K(\mathbb{Z},3) & & \end{array}$$

?? — what's induced twisting ??

E A_∞ ring spectrum, $S \to E$ unit

$BGL_1(E)$ classifies twists of E-theory

$BGL_1(S)$ classifies spherical fibration (stable, of rank zero)

$$X \xrightarrow{V} BSO \xrightarrow{j} BGL_1 S \xrightarrow{\iota} BGL_1 E$$

j-homomorphism

$\Rightarrow$ get induced twist of E-theory

twist $E(X)_{\iota j V} \simeq \tilde{E}(X^V)$ reduced cohomology of Thom space

<u>Example</u> $E = K$

$$\begin{tikzcd}
 & BSpin_c \arrow[d] \arrow[rrr, bend left, "\text{trivial: } E_\infty \text{ spin}^c \text{ orientation on } K\text{-theory}"] & & & \\
X \arrow[ur, dotted] \arrow[r, "V"'] & BSO \arrow[r, "j"'] \arrow[d] & BGL_1 S \arrow[r, "\iota"'] & BGL_1 K \\
 & K(\mathbb{Z},3) \arrow[urr, "+"'] & &
\end{tikzcd}$$

because of E_∞ structure, triviality of $BSpin_c \longrightarrow BGL_1 K$ gives factorization $K(\mathbb{Z},3) \longrightarrow BGL_1 K$

Theorem (Ando Blumberg Gepner Hopkins Rezk)
[construct Thom space for any map, like
$\lambda : X \longrightarrow BGL_1 K \ldots$]

Can form $X^{+\beta w_2 V}$ K-module spectrum

$$\simeq X^V \wedge K$$

$\Rightarrow$ form $K(X)_\lambda := \mathrm{Hom}_{K\text{-mod}}(X^\lambda, K)$

$$\simeq \mathrm{Hom}_{K\text{-mod}}(X^V \wedge K, X)$$

$$\simeq \tilde{K}(X^V)$$

In particular, if $H \in H^3(X, \mathbb{Z})$

$$\Rightarrow X \xrightarrow{H} K(\mathbb{Z}, 3) \xrightarrow{t} BGL_1 K$$

$\Rightarrow$ produce $K(X)_H$

$$\Rightarrow \underline{\underline{K(X)_{\beta w_2 V}}} \simeq K(X^{ijV}) \cong \tilde{K}(X^V)$$

depends only on deg 3 cohomology class: don't need geometry of V!

$$\begin{tikzcd}
 & BString = BO\langle 8\rangle \arrow[d] \arrow[rrr, bend left, "\text{trivial (Ando-Hopkins-Rezk)}"] & & & \\
X \arrow[r] & BSpin \arrow[r, "j"] \arrow[d] & BGL_1 S \arrow[r] & BGL_1(tmf) \\
 & K(\mathbb{Z},4) \arrow[rru] & &
\end{tikzcd}$$

$$\Rightarrow \quad tmf(X)_{\frac{p_1}{2}X} \simeq tmf(X^\vee) = Ell(X^\vee)$$

So RHS completely controlled by degree 4 cohomology class $\frac{p_1}{2}X$.

$$\begin{tikzcd}
D \arrow[r, "\nu"] \arrow[d] & BSpin \arrow[d, "p_1/2"] \\
X \arrow[r, dashed, "H?"'] & K(\mathbb{Z},4)
\end{tikzcd}$$

if have deg 4 class H making this commute get

$$Ell(D) \simeq Ell(D^\vee)_{-H_j} \to Ell(X)_{-H}$$

umkehr map in elliptic cohomology

$$
\begin{tikzcd}
Ell(X) \arrow[r] \arrow[d, dashed, "\text{conjectural}"'] & K(X; \mathbb{Z}[[q]]) \arrow[d, "\cong"] \\
K_{S^1}(LX) \hat{\otimes} \mathbb{Z}[[q]] \arrow[r, "\text{restrict to constant loops}"'] & K_{S^1}(X) \hat{\otimes}_{BS^1} \mathbb{Z}[[q]]
\end{tikzcd}
$$

$\lambda \mapsto 1-q$

$\lambda \in H^4(X)$, use to twist $Ell(X)$

$\rightsquigarrow$ transgress to $\tau(\lambda) \in H^3_{S^1}(LX)$

– can twist diagram to get

$$Ell(X)_\lambda \longrightarrow K(X, \mathbb{Z}[[q]])_{\tau\lambda}$$

G simple simply connected Lie group

$H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$

$$
\begin{tikzcd}
 & \mathbb{Z} & \\
Ell_G(pt)_k \arrow[rr, "\sim"', "\text{Ando, Grojnowski, Ginzburg-Kapranov-Vasserot}"'] & & R_k(LG) \arrow[ul, bend right, "\text{Kac character formula}"'] \\
{}^{k+h}K(G/\!/G) \arrow[u, "ch", "\wr"'] \arrow[urr, bend right, "FHT"'] & &
\end{tikzcd}
$$

Orientations by chromatic level

0	$H\mathbb{Q}$	SO	$BSO \to K(\mathbb{Z}/2,1)$
1	K	$Spin$	$BSpin \to K(\mathbb{Z},4)$
2	Ell	$String$	$BString$
3		stops being this simple - not covers of SO . . .	

$$BString \to BSpin \to BSO$$

$\check{T}$ = cocharacter lattice of $T \subset G$

C elliptic curve

$$Ell_G(pt) = \Gamma\left((\check{T} \otimes C)/W,\ \mathcal{L}^k \right)$$

weighted projective space