Rina Anno - Stability conditions, Springer fibers & braids with multiplied threads.

Note Title 3/28/2009

[Today SL_n]

1. Braid group action on T^*Fl_n (Khovanov-Thomas)
   (ie on $D_{Fl_n}(T^*Fl_n)$)

2. Stability conditions on T^*Fl_n
   (A. - Bezrukavnikov)

3. diagrammatical description of t-structures involved in above (A. - Logrinenko)

$\mathcal{D} = D_{Fl_n}(T^*Fl_n)$ = bounded derived category of Coh T^*Fl_n with support on Fl_n.

Action of affine braid group $ABr_{n} \curvearrowright \mathcal{D}$:

Fl_n has n-1 projections to partial flag varieties $P_{n,k}$: omit k^{th} space

$$Fl_n \xrightarrow{p_k} P_{n,k}$$

$\Rightarrow$ correspondence

$$\begin{array}{ccc}
T^*P_{n,k} \times_{P_{n,k}} Fl_n & \xrightarrow{i_k} & T^*Fl_n \\
\downarrow \pi_k & \searrow & \updownarrow \\
T^*P_{n,k} & \square & Fl_n \\
\updownarrow & \swarrow p_k & \\
P_{n,k} & &
\end{array}$$

i_k : inclusion of a divisor

π_k : $\mathbb{P}^1$ bundle

Action of Br_{n-1}:

generator $\varphi_k = \mathrm{Cone}\{ S_k R_k \longrightarrow id \}$

where $S_k = i_{k*}(\mathcal{E}_k \otimes \pi_k^*) : D_{P_{n,k}}(T^*P_{n,k}) \longrightarrow D$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{D_k}$

$\mathcal{E}_k$ = line bundle on $T^*P_{n,k} \underset{P_{n,k}}{\times} Fl_n$

S_k has right adjoint $R_k = \pi_{k*}(\mathcal{E}_{k+1}^{\vee} \otimes i_k^!)$
& left adjoint $L_k = R_k[2]$

Action of ABr_{n-1}: need also action of root lattice Λ_n

$\mathfrak{h}^* = H^2(Fl_n) \supset \Lambda$: roots $\to$ line bundles, acts on D by tensor product
$= H^2(T^*Fl_n)$

Khovanov-Thomas – these satisfy ABr_{n-1} relations

For each k, $S_k : D_k \longrightarrow D$ are spherical functors : generate twists like spherical objects do. [not faithful..]

$\varphi_k S_k = S_k[-1]$

"ie on image of S_k φ acts by $[-1]$"

On $K^0(T^*Fl_n)$, φ_k is reflection wrt $\langle$span of the image of $S_k\rangle^\perp$.

(note D CY so left & right orthogonals agree)

$\operatorname{Dim} K^0(T^*Fl_n) = n!$

$\dim \operatorname{Im} S_k : \frac{n!}{2}$

$= \dim (\operatorname{Im} S_k)^\perp$: φ_k looks like

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & & -1 \end{pmatrix}$$

Bridgeland's picture:

- t structure with N irreds & Br_N action

$\rightsquigarrow$ open subset in Stab

(simplex w/ N faces)

$Br_N \longleftrightarrow$ flipping the simplex over each face, generate open subset in Stab.

Our setting: exotic t-structure, $\underline{n!}$ simples

Br_{n-1} action : generators $\longleftrightarrow$ n-elt subsets of simples

$$\begin{array}{ccc} \text{Stab} & \overset{\text{closed}}{\hookleftarrow} & \text{Stab}^\circ \\ \downarrow & & \downarrow \\ H^*(T^*Fl_n) & \overset{\exp}{\hookleftarrow} & H^2(T^*Fl_n) \end{array}$$

dim $n!$ (for $H^*(T^*Fl_n)$); dim n (for $H^2(T^*Fl_n)$)

Now on Stab° can follow Bridgeland story to get an open subset $\mathcal{H}$ mapping to $\mathfrak{h}^*_{reg}$

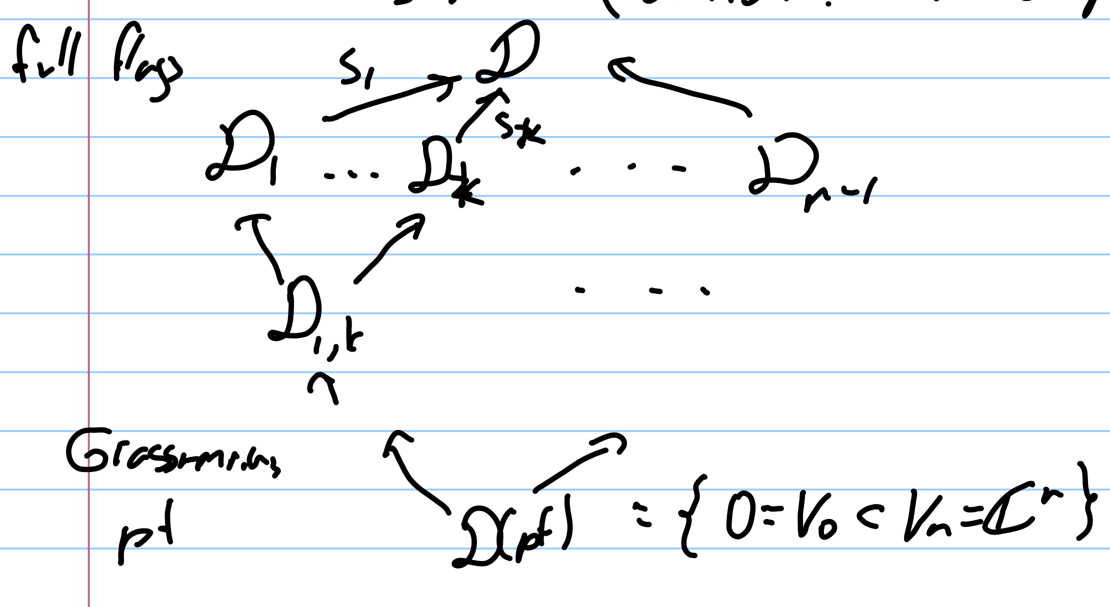
$$\mathcal{H} \downarrow \mathfrak{h}^*_{reg}$$

Property of closed embedding exp which makes this possible: image crosses the reflection walls for φ_k transversally.

The open is then a cover of $\mathfrak{h}^*_{reg}/W$
- deck group is a quotient of AB_{n-1}

The exotic t-structure appears to have a simple combinatorial description (or at least of irreducible objects) diagrammatically

Get a net of functors between T^*'s of partial flags, each with actions: (notation: mark dropped subspace)



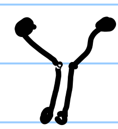
Learn to describe exotic simples step by step from this point:
at each step have exotic t-structures & collection of simple exotic sheaves:
the functors send simple exotics to simple exotics.

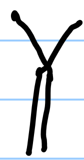
Diagrams: $\mathcal{D}$: • • • • • (n pts on line)

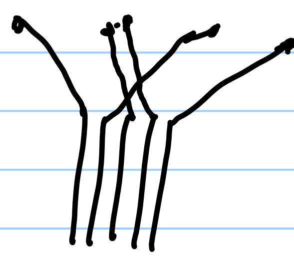
$\mathcal{D}_k$: glue 2 points together
• • • •• • •

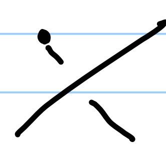
smaller partial flags: ••• • •• • :

$0 = V_0 \subset V_3 \subset V_4 \subset V_5 \subset V_7 = V$

S_k : add an interval  (double line)

$\tilde{S}_k$ 

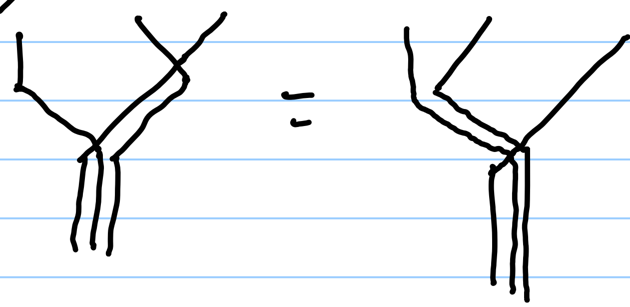
 etc.

+ usual braid pictures 

To each sequence of functors in diagrams get tree like diagram $\longleftrightarrow$ irreducible exotic sheaf

+ isotopy conditions

eg



On Grassmannian: expect twist of Kapranov's exceptional family.

$\mathbb{P}^n$ get Beilinson exceptional sequence $\mathcal{O}(-1), \ldots$

Positive braids are left exact wrt t-structure.