Chris Brav- Projective McKay Correspondence

Note Title 3/28/2009

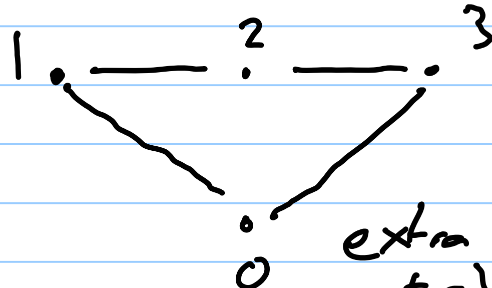
1. McKay correspondence for $\mathbb{P}^1$ (Kirillov Jr.)
2. Also for $T^*\mathbb{P}^1$

McKay: Conjugacy classes of finite subgroups $\tilde{G} \subset SL_2\,\mathbb{C}$ $= SL_2(V)$ $\longleftrightarrow$ affine ADE Dynkin diagrams

$\tilde{G} \longleftrightarrow \Gamma$ vert = irreps of $\tilde{G}$

#arrows(i,j) = $\dim \mathrm{Hom}_G(V_i, V \otimes V_j)$

Ex. $\tilde{G} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle \subset SL_2\,\mathbb{C}$
is $\mathbb{Z}/4$



$\hat{A}_3$

extra node $\longleftrightarrow$ trivial rep

$$\begin{array}{ccc} \tilde{G} & \subset & SL_2 \\ \downarrow & & \downarrow \\ G & \subset & PSL_2 \end{array}$$

G-equivariant coherent sheaves on $\mathbb{P}^1$:

Ex. • $\mathcal{O}(1)$ is $\tilde{G}$-equivariant
$\mathcal{O}(2d)$ is G-equivariant

• $V_i \otimes \mathcal{O}_{\mathbb{P}^1}$ trivial bundles w/ fiber a representation
"half" are G-equivariant (even reps)

Organize with a <u>height</u> function $h : \Gamma \to \mathbb{Z}$

s.t. 1) $h(i) \equiv$ parity of vertex (mod 2)

2) $|h(i) - h(j)| = 1$ for adjacent vertices

$\rightsquigarrow$ quiver Q_h with arrows flowing down in height

& get a list of G-equivariant vector bundles

$F_i = \mathcal{O}(h(i)) \otimes V_i$

with $\dim \mathrm{Hom}_G(F_i^h, F_j^h) = \begin{cases} 0 & i \to j \\ 1 & i \leftarrow j \end{cases}$

<u>Theorem</u> (Kirillov Jr)

For each height h, $\bigoplus F_i^h$

is a tilting complex in $D_G(\mathbb{P}^1)$

... no higher self exts, & generates

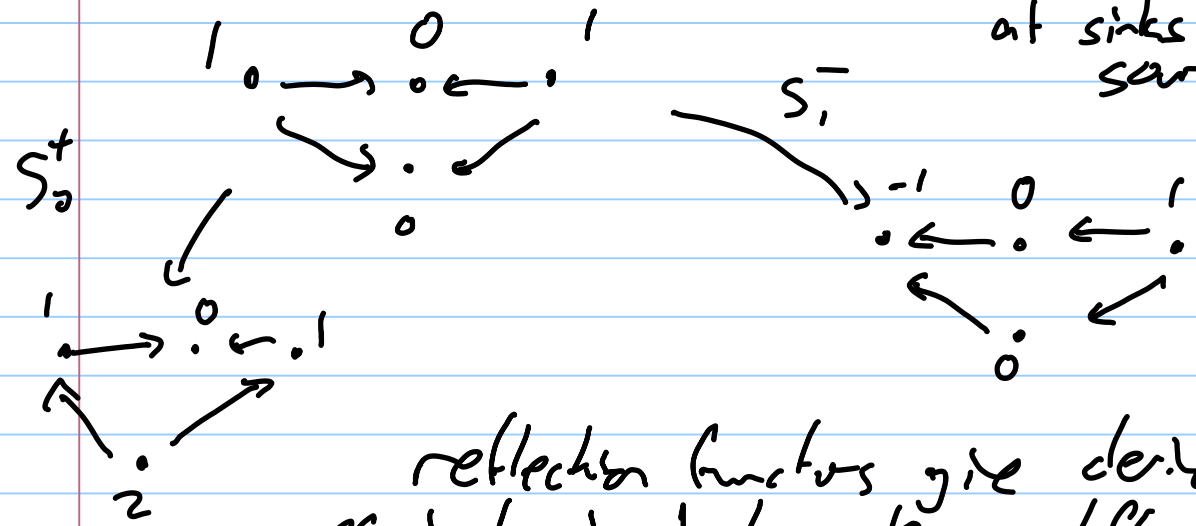
$\Rightarrow D_G(\mathbb{P}^1) \xrightarrow{\sim} D(A_h)$

$A_h = \mathrm{End}_G \bigoplus F_i^h$

Moreover $A_h = \mathbb{C} Q_h^{op}$ path algebra of quiver

Get lots of non-Morita equivalent but derived equivalent algebras.

Relating heights (BGP) : add or subtract 2 at vertices at sinks or sources



reflection functors give derived equivalences between these different quivers. – connects all heights

Thm (Kirillov Jr.)

$$\begin{tikzcd} & D_G(\mathbb{P}^1) \arrow[dl, "\Phi_h"', "\sim"] \arrow[dr, "\Phi_{\hat{h}}", "\sim"'] & \\ D(A_h) \arrow[rr, "\sim", "BGP"'] & & D(A_{\hat{h}}) \end{tikzcd}$$

... proof uses Euler sequence of $\mathbb{P}^1$.

Analog for $T^*\mathbb{P}^1$.

$$\begin{tikzcd} T^*\mathbb{P}^1 \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[u, bend left, "s"] \end{tikzcd}$$

Let $B_h = \mathrm{End}_G(\oplus \pi^* F_i^h)$

<u>Prop</u> $\oplus \pi^* F_i^h$ is a tilting object for $D_G(T^*\mathbb{P}^1)$,

so $D_G(T^*\mathbb{P}^1) \xrightarrow{\sim} D(B_h)$

Moreover $B_n \simeq \Pi_{\bar{Q}_n}$ preprojective algebra of quiver

$\simeq \mathbb{C}\bar{Q}_n / \sum [\alpha^*, \alpha]$ $\quad \bar{Q}$: add arrows in reverse direction

<u>Relating equivalences</u>

Under $D_G(\mathbb{P}^1) \xrightarrow{\sim} D(Q_n)$

$E_i^n \longleftarrow S_i$ simples

Then $\mathcal{E}_i^n = \pi^* E_i^n \longmapsto S_i \in D(B_n)$

<u>Prop</u> The $\mathcal{E}_i^n$ form a Γ-configuration of spherical objects in $D_G(T^*\mathbb{P}^1)$

Upshot: (by Seidel-Thomas) get autoequivalences $T_{\mathcal{E}_i^n}$, spherical twists, generating a braid group $B_\Gamma \hookrightarrow D_G(T^*\mathbb{P}^1)$

Introduce subcategory $\mathcal{D} \subset D_G(T^*\mathbb{P}^1)$ of sheaves supported on zero section.

For each n get a t-structure on $\mathcal{D}$ whose heart $\mathcal{B}_n$ is finite length with simples $\mathcal{E}_i^n$.

Theorem If $i \in Q_h$ is a source, then

$$T_{\mathcal{E}_i^h}(\mathcal{E}_j^h) = \mathcal{E}_j^{s_i^- h}$$ twist of height function

Similarly for sinks $i \in Q_h$

$$T^{-1}_{\mathcal{E}_i^h}(\mathcal{E}_j^h) \simeq \mathcal{E}_j^{s_i^+ h}$$

So $T_{\mathcal{E}_i^h}(\mathcal{B}_h) \simeq \mathcal{B}_{s_i \cdot h}$.

Point: $\mathcal{D}$ is a 2 CY category ... like simplified model of derived category of a K3 surface.

Given a heart of finite length, get a big open piece for space of stability conditions: phase for each simple in $\mathcal{H} \subset \mathbb{C}$.

Spherical twists tell us how to glue nearby ones.

Bridgeland conjectures: $\left(\widetilde{\mathfrak{h}^{reg}/W_{aff}}\right)$ universal cover is a connected component of space of stability conditions

Would like $B_\Gamma \curvearrowright \mathcal{D}$ faithful in particular

Type A: Seidel-Thomas

Other types: progress of Brav - H. Thomas