David Nadler - Character Sheaves

Note Title 3/28/2009

(Urbana, March '09)

Aim: explain how to associate

G reductive$/\mathbb{C}$ $\rightsquigarrow$ 2d categorified TFT χ_G

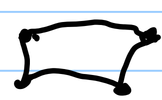
[or partial 3d TFT dims 0,1,2]

Properties: χ_G should know everything about representation theory of G, eg, $G_{\mathbb{R}}$, ...

[Today: with trivial infinitesimal character]

Application: Langlands duality for Lusztig's character sheaves

<u>Toy model:</u> G finite group $\rightsquigarrow$ 2d TFT Z_G

2-morphisms  Σ closed $\longmapsto$ # G-bundles $\{\pi_1 \Sigma \to G\}/\sim$

1-morphisms , $S^1 \longmapsto \mathcal{O}\left(\frac{G}{G}\right)$ class functions

objects $\bullet$ $\longmapsto$ $Rep^{fd}\, G$ category

encodes everything about G-reps, eg can write cobordism that tells you about characters of representations

Everything here is governed by the group algebra $\mathcal{O}(G)$ - "NC Frobenius algebra"

Domain $\qquad\qquad\qquad$ Target

2-category 2Cob $\xrightarrow[\text{symmetric monoidal}]{Z_G}$ 2-category Alg

obj = algebras

1-mor = bimodules

2-mor = maps of bimodules

Lurie (Costello, Kontsevich-Soibelman):

pt $\rightsquigarrow$ $\mathcal{O}(G)$

generates rest of theory freely.

Lie groups

1. G finite $\rightsquigarrow$ G reductive $/\mathbb{C}$

2. Categorify: before $\mathcal{O}(G)$ was an algebra object in $Vect^{fd}$.

Now $Vect^{fd} \rightsquigarrow St$ stable categories (eg dg categories: enhanced derived categories)

algebras: algebras $\rightsquigarrow$ monoidal categories

Want $G/\mathbb{C} \rightsquigarrow$ monoidal category playing role of group algebra "measures on G"

One possibility: $G \rightsquigarrow \mathcal{D}(G)$
D-modules on G

Our task: $B \subset G$ Borel

$$\mathcal{H}_G = \mathcal{D}(B\backslash G/B) \quad \text{Hecke category}$$

Cartoon: 

gauge theory on interval with B-reductions at endpoints

$$\rightsquigarrow \quad pt/B \underset{pt/G}{\times} pt/B$$

Beilinson-Bernstein localization:

$$\mathcal{D}(G/B) \simeq \mathfrak{g}\text{-mod} \quad \left(\begin{array}{l}\text{with trivial}\\ \text{inf.}\\ \text{character}\end{array}\right)$$

Hecke category $\mathcal{H}_G$ gives the natural symmetries of $\mathfrak{g}$-mod by this picture: right action on $\mathcal{D}(G/B)$.

...e.g. Kazhdan-Lusztig: $\mathcal{H}_G$ governs changes of bases for representations

<u>Theorem</u> There is a categorified 2d TFT $\mathcal{X}_G$

$$\begin{array}{rcl}
2Cob & \longrightarrow & \text{Algebras in } St \\
S^1 \times S^1 & \longmapsto & HH_*(Ch_G) \\
S^1 & \longmapsto & Ch_G \text{ unipotent character sheaves} \\
pt & \longmapsto & \mathcal{H}_G \text{ Hecke category}
\end{array}$$

Ch_G : some D-modules on $\frac{G}{G}$, special class functions

Can also think of $\longmapsto$ $\mathcal{H}_G$-modules

What are character sheaves?

Geometric definition : $Ch_G \subset D(\frac{G}{G})$
full subcategory of D-modules whose singular support $\subset$ nilpotent cone
(& unipotent central character)

Ex $G = T$ torus $Ch_T = H^*(T)$-mod
in $D(BT)$
$\simeq H_*(T)$-mod

Main application

Beilinson-Ginzburg-Soergel :
$\mathcal{H}_{G,\mathrm{per}} \longleftrightarrow \mathcal{H}^{\mathrm{mon}}_{G^\vee,\mathrm{per}}$ Langlands dual

$\Rightarrow \begin{cases} \mathcal{X}_G \simeq \mathcal{X}^{\mathrm{mon}}_{G^\vee} & \text{isom of TFTs} \\ Ch_G \simeq Ch_{G^\vee} & \text{[mod periodic]} \end{cases}$

Example 2-periodically $H^*(T)$-mod in $H_*(T)$-mod $\leftrightarrow$ $H^*(T^\vee)$-mod in $H_*(T^\vee)$-mod