Alexei Oblomkov - GW/DT w/ descendents

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<u>GW</u> : X, $\beta \in H_2(X, \mathbb{Z})$ X proj variety

$\overline{M}_{g,n}(X,\beta)$ moduli of stable maps w/ class β

$\downarrow ev_i$ evaluations

X

$\gamma_1, \ldots \gamma_n \in H^*(X)$

$$\Rightarrow \langle \gamma_1 \ldots \gamma_n \rangle^{GW}_{g,\beta} = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_i ev_i^*(\gamma_i)$$

"=" # curves of genus g intersecting $\gamma_i^\vee$ at x_i

$X = \mathbb{P}^n$, $g=0$ all works, no virtual class

$$\dim [M_{g,n}(X,\beta)]^{vir} = -\beta \cdot K_X + (3-\dim X)(g-1) + n$$

<u>DT</u> $C \subset X \subset \mathbb{P}^N$ I_C = ideal of C
$\subset \mathbb{C}[x_0 \ldots x_N]$

$(I_C)_k$ deg k part

<u>Theorem</u> (Hilbert) $\operatorname{codim}_{\mathbb{C}[x_0 \ldots x_N]_k} (I_C)_k$

$= k \cdot \deg(C) + \chi(\mathcal{O}_C)$ $k \gg 0$

$\deg(C) = H \cdot \beta$

Grothendieck: $\mathrm{Hilb}_\chi(X,\beta)$ = moduli of ideals with given Hilbert poly

X projective $\Rightarrow \mathrm{Hilb}_\chi(X,\beta)$ is as well.

$$\begin{tikzcd} \mathcal{I} \arrow[d] & \\ \mathrm{Hilb}\times X \arrow[d, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathrm{Hilb} & X \end{tikzcd}$$

$\mathcal{I}$ universal ideal sheaf

$$ch_k(\gamma) = pr_{1*}\left(ch_k(\mathcal{I})\,\pi_2^*\gamma\right)$$

$$\int_{[\mathrm{Hilb}]^{vir}} \prod_i^n ch_2(\gamma_i) = \langle \gamma_1 \dots \gamma_n \rangle^{DT}_{\chi,\beta}$$

"=" # of curves of Euler char χ intersecting $\gamma_1^\vee, \dots \gamma_n^\vee$.

Ex. $\mathrm{Hilb}_n(\mathbb{C}^3, 0) = \{ I \subset \mathbb{C}[x,y,z],\ \mathrm{codim}\, I = n \}$

dreadful spaces in general!

$\mathrm{Hilb}_0(\mathbb{P}^3, [L]) = Gr(1,3)$

Donaldson-Thomas: $\dim X = 3$ $\exists\, [\mathrm{Hilb}_\chi(X,\beta)]^{vir}$ with nice properties.

$$\dim [\mathrm{Hilb}]^{vir} = -K_X \cdot \beta = \dim [\overline{\mathcal{M}}_{g,0}(X,\beta)]^{vir}$$

<u>GW/DT</u> Let $\langle \gamma_1 \dots \gamma_n \rangle^{GW}_\beta = \sum_g \langle \gamma_1 \dots \gamma_n \rangle^{GW}_{g,\beta} \cdot u^{2-2g}$

(allow disconnected curves)

$$\langle \gamma_1 \dots \gamma_n \rangle^{DT}_\beta = \sum_\chi \langle \text{"} \rangle q^\chi$$

$$\langle \gamma_1, \dots \gamma_n \rangle'^{,GW}_\beta = \frac{\langle \gamma_1 \dots \gamma_n \rangle^{GW}_\beta}{\langle \gamma_1 \dots \gamma_n \rangle^{GW}_0} \longleftarrow \text{degree } 0 \text{ part}$$

same for DT

Conjecture [MNOP]

1. $\langle \gamma_1 \dots \gamma_n \rangle^{DT}_\beta$ is a rational function of q

2. set $q = -e^{iu} \Rightarrow \langle \gamma_1, \dots \gamma_n \rangle^{DT}_\beta = \langle \gamma_1, \dots \gamma_n \rangle^{GW}_\beta$

(expand around $u=0$)

Theorem [MOOP] Conjecture holds for X toric

Change of variables relates to function

$\left(\frac{\sin u}{u}\right)^{2g-2}$ ... appears in random variables, G-W invariants

GW descendants

$\mathbb{L}_i$ = ith cotangent line bundle over $\overline{\mathcal{M}}_{g,n}(X,\beta)$

$c_1(\mathbb{L}_i) = \psi_i$

$$\langle \tau_{k_1}(\gamma_1) \dots \tau_{k_n}(\gamma_n) \rangle$$

$$= \int_{[\overline{\mathcal{M}}]^{vir}} \prod \psi_i^{k_i} ev_i^*(\gamma_i)$$

DT descendants: add rest of Chern character!

$$\langle \sigma_{k_1}(\gamma_1)\cdots\sigma_{k_n}(\gamma_n)\rangle = \int_{\mathrm{Hilb}} \prod ch_{k_i+2}(\gamma_i)$$

Can we relate $\langle\sigma\cdots\rangle^{DT}$ to $\langle\tau\cdots\rangle^{GW}$?

no equality

Conjecture 1. $\langle\sigma_{k_1}(\gamma_1)\cdots\sigma_{k_n}(\gamma_n)\rangle^{DT}_{\beta}$ rational in q

FALSE: but can hope it holds if $\deg\gamma_i>0$ (don't allow descendants of 1's)

What if do have 1's?

$$E_3(q) = q\frac{d}{dq}M(q) \qquad M = \text{MacMahon fn} = \prod_i \frac{1}{(1-q^i)^i}$$

Conjecture 1' If allow 1's

$$\langle\sigma_{k_1}(\gamma_1)\cdots\sigma_{k_n}(\gamma_n)\rangle^{DT}_{\beta} \in \mathbb{Q}(q)[E_3, E_3', \ldots]$$

Write $q=-e^{iu}$.

$\Rightarrow$ Expansion $E_3(u) \sim \frac{2\zeta(3)}{u^3} - \sum \frac{B_{2n+1}B_{2n}}{(2n)!\,(2n+1)}$

we'll drop first (irrational) term.

Theorem (Levine-Pandharipande, J. Li, ..)

$$\langle\ \rangle^{DT}_0 = M(-q)^{\int c_3 - c_1c_2}$$

$\rightsquigarrow$ E_3 appears because of string equation.

<u>Conjecture</u> $(-q)^{r/2} \left\langle \exp\left(\int_X U(ch(\mathbb{I}), t, c_1, c_2, c_3)\right)\right\rangle'_{DT}$

$$= \left\langle \exp\left(\sum_k \sum_j \tau_k(\gamma_j) t_k^j\right)\right\rangle'_{\beta, GW}$$

U: explicit universal function.

where $\gamma_0, \dots, \gamma_r$ basis of cohomology,

don't allow $1 = \gamma_0$.

Define $\mathbb{Q}(q)[E_3, E_3', \dots] \longrightarrow \mathbb{Q}((u))$

$f \in \mathbb{Q}(q) \rightsquigarrow$ expand in $e^{-iu} = q$

$$\left(q\frac{d}{dq}\right)^k E_3 \longmapsto \frac{1}{2}\frac{d^k E_3}{du}$$

if allow 1's above need to use above expansion.

$$U = \sum_k U_k\left(\sum t_k^j \gamma_j\right) + \sum_{k,l} U_{k,l}\left(\sum t_k^j \gamma_j\right)\left(\sum t_l^j \gamma_j\right)$$

+ triple term

$$U_k = ch_{k+2}(\mathbb{I}) + \chi(1,k)\, ch_{k+1}(\mathbb{I})\, c_1(X)$$

$$+ \chi(2,k)\, ch_k(\mathbb{I})\, c_1^2(X)$$

$+ \dots$

where $\chi(m,k) = [t^m] \dfrac{1}{\prod^k (t \cdot i}$

Proven for:

- $Y \xrightarrow{\mathbb{P}^1} K3$
- $\mathbb{P}^3$ _ degs 1 & 2.