Note Title 3/27/2009

Dima Shklyarov - Equivariant TFT & Representation Theory

G - fixed finite group
K - fixed ground field.

A K-valued class function $f: G \to K$ is a function with $f(ghg^{-1}) = f(h)$

Such functions arise as characters of K-linear representations of G

A K-valued <u>n-class</u> function is a function on n-tuples of pairwise commuting elements of G with

$$f(gh_1g^{-1}, gh_2g^{-1}, \ldots gh_ng^{-1}) = f(h_1, \ldots h_n)$$

Such functions appear in nature: for some interesting generalized cohomology theories, classes in $h^*(BG)$ are represented by n-class functions (Hopkins-Kuhn-Ravenel).

Q: What is the representation theory underlying these?

Ganter - Kapranov (2007):
n-class functions should arise as characters of G-actions on higher K-vector spaces
... study n=2 in detail

What are 2-vector spaces?

For my purposes [S.]:

2-vector space = K-linear additive category.

A G-action on a 2-vector space is given by

$\forall g \in G$ $\rho(g)$ functors, &

natural isomorphisms $\rho(g) \circ \rho(h) \to \rho(gh)$

These have to satisfy some natural conditions (associativity etc)

The categorical trace

$$Tr_\rho = \bigoplus_{g \in G} Tr_\rho(g),$$

$$\left[Tr_\rho(g) = Nat(Id, \rho(g)) \right]$$ .... a G-equivariant G-graded vector space

The 2-character of ρ is

$$\chi_\rho(g,h) = tr\left(h : Tr_\rho(g) \to Tr_\rho(gh)\right)$$

a 2-class function.

Examples 1. X = finite G-set, $\mathcal{F}(X)$ = algebra of K-valued fns on X

$$Mod_X = \text{f.d. } \mathcal{F}(X)\text{-modules}$$

$$\rho(g) : M \longmapsto M \otimes_{\mathcal{F}(X)} \mathcal{F}_g(X)$$

$\mathcal{F}_g(X) = \mathcal{F}(X)$ with g-twisted bimodule structure.

$$\mathcal{F}_g(X) \otimes_{\mathcal{F}(X)} \mathcal{F}_h(X) \xrightarrow{\sim} \mathcal{F}_{gh}(X)$$

$$Tr_\rho(g) \simeq \{ f(x) \in \mathcal{F}(X) : \operatorname{supp} f(x) \subset X^g \}$$

$$\chi_\rho(g,h) = \# X^{g,h}$$

2. Fix a K^*-valued 2-cocycle on G

$c : G \times G \longrightarrow K^*$, normalized : $c(1,-) = c(-,1) = 1$

$\Longrightarrow$ G-action on $Vect_K$:

$\rho(g) = Id \quad \forall g \in G$

$$\rho(g) \circ \rho(h) \xRightarrow{\cdot c(g,h)} \rho(gh)$$

$$\chi_\rho(g,h) = \frac{c(h,g)\, c(gh, h^{-1})}{c(h, h^{-1})}$$

3. $G = \pm 1$, char $K \neq 2$

Ob A = pairs (V, E_V) $V \in Vect_K$

$E_V \in End\, V$, $E_V^2 = 0$

$\rho(1) = Id \quad \rho(-1) : (V, E_V) \longmapsto (V, -E_V)$

$$\chi_\rho(1,1) = 2 \qquad \chi_\rho(1,-1) = 0$$

$$\chi_\rho(-1,1) = 1 \qquad \chi_\rho(-1,-1) = -1$$

$SL_2\mathbb{Z}$ acts on pairs of commuting elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (g,h) \longmapsto (g^a h^b, g^c h^d)$$

The 2-character in example 1 is $SL_2\mathbb{Z}$ invariant

" " 2 also also

" " 3 is <u>not</u>.

In Ex 1 & 2 the categories are modules over strongly separable algebras B.

Separable algebra: B is projective as B-bimodule.

Strongly separable: separable &

$$(a,b) \longmapsto \mathrm{tr}_B(L(a)L(b))$$

is nondegenerate.

In char 0: same as semisimple algebra.

We will call the category of f.d. modules over a strongly separable algebra a <u>dualizable</u> 2-vector space.

Thm 1 The 2-character of an arbitrary dualizable 2-representation of G is $SL_2\mathbb{Z}$ invariant.

This fact should admit a field-theoretic explanation.

The right class of QFT was defined by Turaev: equivariant TFT

Essentially, an equivariant TFT is a symmetric monoidal functor from the category of principal G-bundles over closed oriented 1-manifolds, with morphisms = principal G-bundles over 2d cobordisms, to vector spaces.

Turaev: equivariant TFT $\Longleftrightarrow$ crossed G-algebra

Roughly speaking, a crossed G-algebra $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ is a unital commutative Frobenius algebra in the braided monoidal category of G-equivariant G-graded vector spaces satisfying

Axiom 1 $g|_{\mathcal{C}_g} = Id$

Axiom 2 (torus axiom) for all $g, h \in G$ & $c \in \mathcal{C}_{hgh^{-1}g^{-1}}$,

$$Tr_{\mathcal{C}_h}(L(c)\cdot g) = Tr_{\mathcal{C}_g}(h^{-1}L(c))$$

Theorem 1 follows from the following fact:

Theorem 2. The categorical trace of an arbitrary dualizable 2-representation carries an equivariant TFT structure.
[observed first by Moore-Segal]

Thm 2 ⟹ Thm 1: Fix $C = \bigoplus_{g \in G} C_g$

$\chi_\rho(g,h) = tr(h : C_g \to C_g)$

$\chi_\rho(g^a h^b, g^c h^d) = \chi_\rho(g, h)$.

Enough to prove that $\chi_\rho(gh, h) = \chi_\rho(g, h)$
$\chi_\rho(h^{-1}, g) = \chi_\rho(g, h)$.

Follow from axioms 1, 2.

One could try to replace additive categories by (enhanced) derived categories & prove that the categorical trace of a G-action on a saturated CY category carries a structure of G-equivariant CohTFT, as defined by Jarvis, Kauffmann & Kimura.