Note Title 3/27/2009

Matt Szczesny - Moduli problems & localization functors for orbifold theories

<u>Motivations</u> CFT $\Rightarrow$ Sheaves on $\mathcal{M}_{g,n}$ via localization functors & Bun_G

$\mathcal{A}$ chiral algebra $\Longrightarrow$ sheaf $\mathcal{F}$ of coinvariants

$$\mathcal{F}|_{(X,x_i)} = \mathcal{H}((X, x_i, M_i))$$

eg $\mathcal{A} = L_k(\mathfrak{g})$ basic rep of Kac-Moody at level $k \in \mathbb{Z}$:

$$\mathcal{H}(X, x, L_k(\mathfrak{g})) = \text{nonabelian theta fns}$$
$$H^0(Bun_G(X), \Theta^k)$$

Extend these constructions to orbifold CFT

<u>Plan</u>: 1. chiral algebras
2. coinvariants
3. orbifold models & moduli occurring there

1. Chiral algebras (Beilinson - Drinfeld)

X smooth curve / $\mathbb{C}$

<u>Def</u> A chiral algebra $\mathcal{A}$ is a right $\mathcal{D}$-module on X with a bracket

$$\{,\} : \mathcal{A} \boxtimes \mathcal{A}(\infty\Delta) \longrightarrow \Delta_! \mathcal{A}$$

skew symmetric, satisfies Jacobi identity

Examples $\cdot$ $\mathcal{A} = \Omega^1_X$

$$\{\ \}: \Omega^1_X \boxtimes \Omega^1_X(\infty\Delta) \to \Omega^2_X(\infty\Delta)/\text{reg} \cong \Delta_! \Omega^1_X$$

More interesting examples come from reps of affine Lie algebras.

- $\mathcal{A}(\mathfrak{g}, k)$: fibers look like

$$V_{\mathfrak{g},k} = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}}^{\hat{\mathfrak{g}}} \mathbb{C}_k$$

Coinvariants de Rham sheaf $h(\mathcal{A}) = \mathcal{A} / \mathcal{A} \cdot T_X$

sheaf of Lie algebras.

Given a collection of $\mathcal{A}$-modules $M_1, \ldots M_n$ at points $x_1, \ldots x_n$

$$\Gamma(h(\mathcal{A}), X \setminus \{x_i\}) \curvearrowright \mathbb{M} = \bigotimes h(M_i)$$

$$H(\mathcal{A}, X, x_i, M_i) = \mathbb{M} / h(\mathcal{A}) \cdot \mathbb{M}.$$

Ex $\mathcal{A} = \mathcal{A}(\mathfrak{g}, k)$

$\mathcal{A}$-modules = smooth $\hat{\mathfrak{g}}_k$-modules.

$$\mathfrak{g}_{out} = \mathrm{Fun}(X \setminus x_i) \otimes \mathfrak{g}$$

Fixing a parameter t_i at x_i an element of $\mathfrak{g}_{out}$ gives me an elt. of $\hat{\mathfrak{g}}$ at each pt

Given $\hat{\mathfrak{g}}_k$-modules $M_1, \ldots M_n$ at $x_1, \ldots x_n$
$\mathbb{M} = \otimes M_i$ is a $\mathfrak{g}_{out}$ module
& $H(A(\mathfrak{g},k), X, x_i, M_i) = \mathbb{M}/\mathfrak{g}_{out}\mathbb{M}$.

Q: What happens when we move X, x_i?
Coinvariants form sheaf with projectively flat connection.

In some cases, can twist by G-bundles
$\rightsquigarrow$ sheaf over Bun_G w/proj flat connection.

Orbifold CFT: Γ finite group
Look at chiral algebras over orbicurves
$[X/\Gamma] \longleftrightarrow$ Γ-equivariant chiral algebras on X. A-modules supported at stacky points $\longleftrightarrow$ "twisted modules"

A-modules at stacky points $\longleftrightarrow$ modules for $\hat{\mathfrak{g}}^{\gamma}_k$ twisted affine algebra:

$x \in X$, Stabilizer $Stab_x = \langle \gamma \rangle$
γ acts on $\mathfrak{g}$, $L\mathfrak{g}^{\gamma} = (\mathfrak{g} \otimes \mathbb{C}((t)))^{\langle \gamma \rangle}$

Can formulate notion of coinvariants as before, where we have twisted modules of stacky points:

$$\mathfrak{g}_{out} = \left[\mathrm{Fun}(X \setminus \Gamma \cdot x_i) \otimes \mathfrak{g} \right]^{\Gamma}$$

$\exists$ moduli space (Abramovich-Corti-Vistoli) $\mathcal{M}_{g,n}^{\Gamma}$ of curves

$$\begin{array}{c} Y \ni y_1, \ldots, y_n \\ \downarrow \\ X \ni x_1 \ldots x_n \end{array},$$

Γ-principal bundle off given points

<u>Theorem</u> Coinvariants form a sheaf with projectively flat connection, more precisely a $D_{\pi^* \mathcal{L}^{c/2}}$-module on $\mathcal{M}_{g,n}^{\Gamma}$ $\leftarrow$ universal curve

Here

$$\begin{array}{ccc} \mathcal{M}_{g,n}^{\Gamma} & \longrightarrow & \mathcal{M}_{g,n} \leftarrow \mathcal{L} = \text{det line bundle} \\ (Y \to X) & \longmapsto & X \end{array}$$

G-bundle setting.

P principal G-bundle, Γ equivariant in some sense:

$$\mathrm{Bun}_G^{\pi,\phi} = \left\{ P, f_\gamma{}_{\gamma \in \Gamma} \quad \begin{array}{l} P \ G\text{-bundle} \\ \phi : \Gamma \to \mathrm{Aut}\ G \\ f_\gamma : \gamma^* P \longrightarrow P \times_{G,\phi} G \end{array} \right\}$$

Theorem Coinvariants for $A(\mathcal{P}, \mathfrak{g}, t)$ form a sheaf w/ proj. flat connection on $Bun_G^{\Gamma, \phi}$.

Example $G = \mathbb{C}^*$, $\Gamma = \mathbb{Z}/2$, $\gamma(t) = \frac{1}{t}$

$Bun_{\mathbb{C}^*}^{\mathbb{Z}/2, \phi}$ = Prym variety : odd part of Jacobian of cover.

Would like a geometric picture!
isomorphism between these coinvariants & twisted nonabelian Θ-functions
$\rightsquigarrow$ try to degenerate, go to boundary & get Verlinde type formula ..