Roman Bezrukavnikov - Local Geometric Langlands

Note Title 1/16/2007

Icm: math.RT/0604445 (w/ Arkhipov, Mirković, Rumynin, Anno)

Study local Langlands categories (Iwahori invariant part)

Local Langlands conjecture:

representation of Galois group of a local field to algebraic group ^LG $\rightsquigarrow$ representation of $G(F)$, F nonarchimedean local field

Geometric theory: (categorification)

^LG-local systems on a punctured formal disc $\rightsquigarrow$ Category with an action of $G(\mathbb{C}((t)))$

discussed by Frenkel-Gaitsgory: suggest constructing these categories via representations of affine Kac-Moody algebras.

A regular singular local system $\longleftrightarrow$ monodromy: conjugacy class in ^LG.

We'll assume this conjugacy class is unipotent — most interesting case (others reduce to it).

Corresponding Iwahori-equivariant part of the desired $G(\mathbb{C}((t)))$ — category should be categorification of a fin dim vector space. This is the main character today.

Philosophical Question (half serious): explain ubiquity of (affine) braid groups. Appears in at least 2 contexts:

1. topological (as a fundamental group)
2. Hecke algebras

Get from one to another by composition of geometric Langlands & mirror symmetry!

$$1. \underset{\substack{\text{geometric}\\ \text{Langlands}}}{\longleftrightarrow} \text{geometry of } {}^L G \underset{\substack{\text{resembles}\\ \text{mirror}\\ \text{symmetry} \ldots \text{ Bridgeland-i}}}{\longleftrightarrow} 2$$

1. Recall: braid group $B_n = \pi_1\left(\mathbb{C}^n \setminus \Delta / S_n\right)$

$B_{aff,n} = \pi_1\left((\mathbb{C}^*)^n \setminus \Delta / S_n \right)$

– think of these numbers as eigenvalues of a matrix:

$B_{aff}(G) = \pi_1\left(G^{rs} / \text{conjugation} \right)$

regular semisimple conjugacy classes

(G not simply connected this is <u>extended</u> braid group)

2. affine Hecke algebra

$\mathcal{H} = \mathbb{C}[I \backslash G(F) / I]$ F nonarchimedean local field with residue field $\mathbb{F}_q$

$\mathcal{H}$-mod $\longleftrightarrow$ $G(F)$-mod generated by Iwahori invariants :

classified by Kazhdan–Lusztig, part of local Langlands conjecture

$\mathbb{C}[B_{aff}] \twoheadrightarrow \mathcal{H}$:

first $\quad B_{aff} \twoheadrightarrow W_{aff} = W \ltimes \Lambda^\vee$ coweight lattice

$$\left(\begin{array}{ccc} B_{aff} & \longrightarrow & W_{aff} \\ \cup & & \cup \\ B(G) & \longrightarrow & W \end{array} \quad \text{finite braid \& Weyl groups} \right)$$

On level of sets have a canonical section $W_{aff} \longrightarrow B_{aff}$, $w \mapsto \tilde{w}$

$$B_{aff} = \langle \tilde{w} \mid w \in W_{aff} \rangle \Big/ \begin{array}{l} \widetilde{w_1 w_2} = \tilde{w}_1 \tilde{w}_2 \\ \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \end{array}$$

length fun

$$W_{aff} = B_{aff} / \tilde{s}_\alpha^2 = 1$$

$$\mathcal{H} = \mathbb{C}[B_{aff}] / (\tilde{s}_\alpha + 1)(\tilde{s}_\alpha - q) = 0$$

[follows from Iwasawa decomposition of $G(F)$]

Appearance of B_{aff} in geometry of ${}^L G$:

motivated by Springer theory, action of W on cohomology of Springer fibers

Let $\mathcal{B} = {}^LG/{}^LB$ flag variety

$\tilde{\mathcal{N}} = T^*\mathcal{B}$

$\downarrow \mu$ moment map [Springer map]

$\mathcal{N} \subset ({}^L\mathfrak{g})^* \simeq {}^L\mathfrak{g}$ cone of nilpotent elements

Springer fiber: $e \in \mathcal{N}$, $\mathcal{B}_e = \mu^{-1}(e)$

Springer: $W \circlearrowright H^*(\mu^{-1}(e))$.

Can extend to action of $W^{aff}(G)$

& deform to an action of $\mathcal{H}(G)$

--- Hecke of dual to ${}^LG$.

--- lattice of translations acts by multiplication by Chern character of a line bundle: labelled by weights of ${}^LG$, ie coweights of G.

$\Lambda^\vee \ni \lambda$ weight of ${}^LG$ sends $h \mapsto h \cdot ch(L_\lambda)$

$\cap$
W_{aff}

L_λ line bundle on $\mathcal{B}$

Theorem We have an action of $\mathcal{B}_{aff}$ on
$D^b Coh(\widetilde{\mathcal{N}})$, & $D^b Coh({}^L\widetilde{\mathfrak{g}})$

${}^L\widetilde{\mathfrak{g}}$ = universal extension of trivial bundle by $T^*\mathcal{B}$
$= \{ b \in \mathcal{B},\ x \in {}^L b \} \supset \widetilde{\mathcal{N}}$.

Properties This action is "geometric over ${}^L\mathfrak{g}$"
... comes from integral kernels
ie can define a monoidal structure on
$D^b Coh({}^L\widetilde{\mathfrak{g}} \times_{{}^L\mathfrak{g}} {}^L\widetilde{\mathfrak{g}})$ & $D^b Coh(\widetilde{\mathcal{N}} \overset{L}{\times}_{{}^L\mathfrak{g}} \widetilde{\mathcal{N}})$

[... in latter fiber product have higher tors,
so should be considered as a dg scheme.]
These monoidal categories act on
$D^b({}^L\widetilde{\mathfrak{g}})$ & $D^b(\widetilde{\mathcal{N}})$.

$\mathcal{B}_{aff} \longrightarrow$ invertible objects in these
monoidal categories
These actions factor through the ${}^L G$-__equivariant__
derived categories of coherent sheaves.

This yields an action on base changed categories $D^b Coh(Y \times_{^L\mathfrak{g}} \tilde{N})$.

Nice case (no higher tors): Y is transversal slice to an orbit (Slodowy)

For slice to subregular orbit get braid group actions on Coh of ALE spaces $\widetilde{\mathbb{C}^2/\Gamma}$. (or stack $[\mathbb{C}^2/\Gamma]$ by McKay correspondence)

Geometric observation:

$$\begin{array}{ccc} {}^L\tilde{\mathfrak{g}} & \supset & {}^L\tilde{\mathfrak{g}}^{reg} \\ \downarrow & & \downarrow W \\ {}^L\mathfrak{g} & \supset & {}^L\mathfrak{g}^{reg} \end{array}$$

ramified Galois covering with Galois gp W

For $w \in W$ let Γ_w be the closure of the graph of w in ${}^L\tilde{\mathfrak{g}} \times_{^L\mathfrak{g}} {}^L\tilde{\mathfrak{g}}$

[let $\Gamma_w' = \Gamma_w \cap (\tilde{N} \times_N \tilde{N})$]

We have $\Lambda^\vee \longrightarrow B_{aff}$ $\hookleftarrow W_{aff} \swarrow$

lattice lifts to B_{aff} as monodromies around the tori

B_{aff} is spanned by these operators θ_λ & the finite Weyl group

The action of B_{aff} is defined by

$$\tilde{w} \longmapsto \mathcal{O}_{\Gamma_w} \quad , \quad \theta_\lambda \longrightarrow \text{twist by } \mathcal{L}_\lambda$$

(new description of old action)

— Contains algebro-geometric properties of Γ_w's, e.g. Cohen-Macaulay.

Standard categorification of $\mathcal{H}$ (in which B_{aff} give invertible objects):

affine flag variety $\mathcal{F}\ell = G((t))/I \hookleftarrow$ Iwahori as algebraic group

$D_I(\mathcal{F}\ell)$ is a monoidal category, categorifying $\mathcal{H}$

<u>Theorem</u> $D_I(\mathcal{F}\ell) \simeq D^b\mathrm{Coh}_{{}^LG}(\tilde{\mathcal{N}} \times_{{}^L\mathfrak{g}} \tilde{\mathcal{N}})$

(as monoidal categories)

categorification of Kazhdan-Lusztig theorem

$$\begin{array}{ccc} D_I(\mathcal{F}\ell) & \simeq & D^b\mathrm{Coh}_{{}^LG}(\tilde{\mathcal{N}} \times_{{}^L\mathfrak{g}} \tilde{\mathcal{N}}) \\ \circlearrowright & & \circlearrowright \\ D^b\left(\begin{array}{c}\text{Whittaker} \\ \text{sheaves on } \mathcal{F}\ell\end{array}\right) & \simeq & D^b\mathrm{Coh}_{{}^LG}(\tilde{\mathcal{N}}) \end{array}$$

antispherical module

Whittaker sheaves are an abelian category

$\rightsquigarrow$ get t-structure on $\mathrm{Coh}_{{}^LG}(\tilde{\mathcal{N}})$

which is <u>local</u> wrt the Springer map:

Can use the equivalence to transfer Whittaker perverse sheaves on $\mathcal{F}\ell$ to a t-structure on $D^b\mathrm{Coh}_{\mathcal{B}_e}(\tilde{\mathcal{N}})$ sheaves supported on Springer fiber

Thus for $e \in \mathcal{N} \Longrightarrow$ abelian category $\mathcal{A}_e$ with an equivalence $D^b \mathcal{A}_e \simeq D^b \mathrm{Coh}_{\mathcal{B}_e} \tilde{\mathcal{N}}$.

Can also basechange, replacing $\tilde{\mathcal{N}}$ by the resolution of a slice: $\tilde{\mathcal{N}} \times_{\mathfrak{g}} Y$

e.g $\widetilde{\mathbb{C}^2/\Gamma}$.

In general $\mathcal{A}_e$ appears in representation theory!

[BMR]: over a field of positive characteristic it can be identified with a category of Lie-modules

over $\mathbb{C}$ it can be identified with modules over quantum groups at roots of unity or category $\mathcal{O}$ at the critical level for $\hat{\mathfrak{g}}$.

Similarity to Bridgeland's theory:

stability conditions on triangulated categories

two equivalent defs: • categories of stable objects indexed by $\mathbb{R}$

• (t-structure, central charge $Z: K^0 \to \mathbb{C}$) + conditions

When we deform Z the t-structure deforms canonically.

Set of stability conditions is a variety ... in certain cases looks like covering of a domain in K_0
(K3: complement of root hyperplanes)

Group of deck transformations act by autoraphism.

t-structures above give family of stability conditions / complement to root hyperplanes, & braid group action $\longleftrightarrow$ covering condition.

Case of $\widetilde{\mathbb{C}^2/\Gamma}$: Bridgeland.

Above construction generalizes his work to transversal slice to any nilpotent.

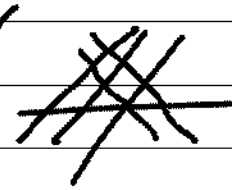
<u>More precisely</u>:

Let $\mathcal{D}_e = D^b \, Coh_{B_e}(\tilde{\mathcal{N}} \underset{\mathfrak{g}}{\times} Y_e)$

Y_e = transversal slice to the orbit of $e \in \mathcal{N}$

$$H^*(\mathcal{B}) \longrightarrow K^0(\mathcal{D}_e)^*$$
$$\|$$
$K^0(\tilde{\mathcal{N}})$: use pairing between the two.

$$\exp: H^2(\mathcal{B}, \mathbb{C}) \longrightarrow H^*(\mathcal{B})$$
$$\|$$
$\mathfrak{h}^*_{\mathbb{C}} \supset \mathfrak{h}^*_{\mathbb{C}} \setminus H_{\check{\alpha}, n}$ complement of affine coroot hyperplanes

( $\mathfrak{sl}_3$)

$$\mathfrak{h}_{\mathbb{C}} \supset \mathfrak{h}_{\mathbb{R}} \supset \Delta$$

Claim (w/ R. Anno). The t-structure, equipped with $Z = \exp(a)$ ($a \in$ fundamental alcove) is a stability condition

2. This map from fund. alcove $\longrightarrow$ stability conditions extends to an embedding of a covering of $\mathfrak{h}_{\mathbb{C}} \setminus H_{\alpha, n}$ into Stab,

compatibly with the action of the affine braid group.

3. The t-structure described before can be characterized by simple properties of that structure (monotonicity & normalization that $R\Gamma$ exact on 0-alcove)

(Note $B_{aff} = \pi_1 \left(\mathfrak{h}_{\mathbb{C}}^{\times} \setminus H_{\alpha,n} / W_{aff} \right)$

[Seidel-Thomas: faithfulness of (nonaffine) braid group actions in type A ...]