Olivier Biquard: Wild Nonabelian Hodge Theory

Note Title 1/11/2007

$p \in X$ Riemann surface

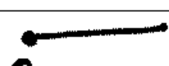
	parabolic Higgs bundles		filtered local systems
residue	$-\frac{1}{2}(\beta + i\gamma)$	monodromy	$e^{-2\pi(\alpha - i\gamma)}$
weight	α		$-\beta$

Here $G^{\mathbb{C}}$ complex, G compact $T \subset G$

$\alpha, \beta, \gamma \in \underline{t}$ (really $\alpha \in \mathfrak{t}/\Lambda = T$

[Higgs Parabolic structure is induced by α]

Filtered local system:

1. $G^{\mathbb{C}}$ local system on $X \setminus p$
2. If we trivialize along a ray into p 
   choose $P \subset G^{\mathbb{C}}$ parabolic
   & a dominant character β of P
   (encodes behavior of a harmonic metric)

Gl_n: just a filtration by subbundles labelled by weights
s.t. connection preserves it

Gukov-Witten have in addition $\eta \in {}^L T$
exchanged with α by S-duality.

Nilpotent residue: Higgs field Y or monodromy e^Y

Model: $\varphi = Y \frac{dz}{z}$ $\quad h = (\ln|z|)^{H}$

Y goes into an sl_2 triple (H, X, Y)
$H = H^*$, $Y = X^*$ (i.e. $su_2 \to \ldots$)

Nilpotent part gives small perturbation
– can superimpose on solutions in case of semisimple residue

residue $\longrightarrow -\frac{i}{2}(\beta + i\gamma + Y)$ $\quad$ monodromy $\to$
$e^{-2\pi(\alpha - i\gamma)} + Y$

Moduli spaces $\mathcal{M}(\alpha, \beta, \gamma; \sigma)$ $\quad \sigma \in C(\alpha, \beta, \gamma)$ centralizer

Higgs POV: Let's first suppose $\alpha = 0$.

$\mathcal{M}(0, \beta, \gamma; \sigma) =$ $G^{\mathbb{C}}$-bundles with Higgs field
$\varphi \in H^0(\Omega^1_{\log D} \otimes E(\mathfrak{g}^{\mathbb{C}}))$

s.t. $\mathrm{Res}\,\varphi$ conjugate to $-\frac{i}{2}(\beta + i\gamma + \gamma)$

--- hyperkähler moduli space

2. $\alpha \neq 0$ $\quad (E, \varphi)$ $G^{\mathbb{C}}$ parabolic $G_{\mathbb{C}}$-bundles

$\varphi \in H^0(\Omega^1_{\log D} \otimes E(\log^{\mathbb{C}}))$.

$$\mathrm{Res}_x\, \varphi \in \mathfrak{p}$$

Write $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n} \qquad \mathfrak{l} = C(\alpha)$

Condition: Projection to $\mathfrak{l}$ of $\mathrm{Res}_x\,\varphi \in L$-orbit of $\beta + i\gamma + \gamma$

(no condition on $\mathfrak{n}$ part).

if α Regular $\Rightarrow \mathfrak{l} = \mathfrak{t}^{\mathbb{C}}$ & we're just fixing the residue $\mathrm{Res}_x\,\varphi = \beta + i\gamma$

$\sigma : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ fixed, γ nilpotent.

<u>Dimension</u> : $G^{\mathbb{C}}$ simple $\Rightarrow$

$$\dim \mathcal{M}(\alpha, \beta, \gamma; \sigma) = 2(g-1)\dim G^{\mathbb{C}} + \dim G^{\mathbb{C}}/H$$

$$H = \mathrm{Stab}(\alpha, \beta, \gamma; \sigma)$$

ex. $\alpha = 0$ $\qquad G^{\mathbb{C}}/H$ = orbit of $\beta + i\gamma + Y$

ex. α regular, $\beta = \gamma = Y = 0$ $\qquad G^{\mathbb{C}}/H$ = regular orbit

$G^{\mathbb{C}} / T^{\mathbb{C}}$

"Special case": S^1-invariant solutions
on the disc + fixed trivialization on boundary.
look like $d + Y \frac{d\theta}{2}$ $\quad Y$ in fixed orbit

$\Longrightarrow$ get exactly the coadjoint orbit of Y,
which is hyperkähler (Kronheimer, Biquard)

— can glue these solutions in to unramified
moduli space to produce ramified solutions.

Varying σ get stratified singular space.

Hol. Poisson moduli space

$$\mathcal{M}(\alpha) = \bigcup_{\beta, \gamma, \sigma} \mathcal{M}(\alpha, \beta, \gamma; \sigma)$$

$\downarrow$ Hitchin $\qquad$ symplectic leaves are hyperkähler

$$\bigoplus H^0(X, K^{d_i}(d_i D))$$

Completely integrable

Interesting leaf: $\alpha \neq 0$, $\beta = \gamma = Y = 0$

$\mathcal{M}(\alpha, 0, 0; 0) \supset T^* \mathcal{M}^{par}(\alpha)$ moduli of parabolic bundles.

Condition: $\mathrm{Res}_x \varphi \in \mathfrak{n}$

For other β, γ, σ they are still of the dimension of $T^* \mathcal{M}^{par}$ but don't contain T.

Wild ramification

Meromorphic connections with higher order poles

Local model:

$$d + A_n \frac{dz}{z^n} + A_{n-1} \frac{dz}{z^{n-1}} + \cdots + A_1 \frac{dz}{z} + \cdots$$

Regular case: can always locally gauge transform connection $\nabla = d + A_1 \frac{dz}{z}$

The analogous statement fails in irregular case

Formally can always gauge away all holomorphic terms -- ie to form $d + A_n \frac{dz}{z^n} + \cdots + A_1 \frac{dz}{z}$.

Wild non-abelian Hodge theorem : Biquard-Boalch

Assumptions : $A_n, A_{n-1}, \ldots A_2$ commuting
<u>semisimple</u> & A_1 commutes with the A_i.
[eg if A_n reg ss this can always be achieved]
Moduli space : fix formally the polar part

Higgs side: look at Higgs bundle (E, φ)
$\varphi = T_n \frac{dz}{z^n} + \ldots$ with same assumptions
on the T_i as the A_i
Moduli space : fix the polar part.

Results: 1. Can produce harmonic metrics, giving a 1:1 correspondence between stable Higgs bundles & stable meromorphic connections

$i > 1$ $\quad T_i = \frac{1}{2} A_i$

$i = 1$ same as in tame case

2. The resulting moduli spaces are hyper-kähler.

<u>Example</u> $\mathbb{P}^1$ with order 2 pole at ∞, order 1 at 0.

Local model $d + A_0\, dz + \Lambda \frac{dz}{z}$ as $z \to \infty$

$d + B_0 \frac{dz}{z}$ as $z \to 0$

The moduli space $\mathcal{M} \supset$ open subset $\mathcal{M}^*$ consisting of meromorphic connections whose underlying bundle is trivial:

$d + A_0\, dz + B \frac{dz}{z}$ A_0 reg ss, Λ diagonal

$\Rightarrow$ B must be in orbit of B_0 & diagonal part of B is Λ.

Symmetries $= T^{\mathbb{C}}$.

So $\mathcal{M}^* = \mathcal{O}(B_0) /\!/_{\Lambda}\, T^{\mathbb{C}}$ hyperkähler reduction

Boalch: $\mathcal{M}$ is a leaf in Poisson Lie group dual to G