Edward Frenkel : Ramified Geometric Langlands

Note Title 1/16/2007

Explain work of Gaitsgory - Frenkel & its relations to Kapustin - Witten & Gukov - Witten

X smooth projective curve / $\mathbb{C}$, $G/\mathbb{C}$ reductive, simple, 1-connected

${}^LG$ = Langlands dual

$E = (\mathcal{F}, \nabla)$ "local system" : $\mathcal{F}$ holomorphic ${}^LG$-bundle on X, ∇ holomorphic or meromorphic connection on X.

Goal: $E \rightsquigarrow$ category Aut_E of Hecke eigensheaves on moduli $\mathrm{Bun}_{G,(y_i),?}$ of G-bundles on X with extra structures at the marked points $y_1, \dots, y_n$ (poles of ∇)

- e.g. parabolic structure (in case of simple poles of ∇)

or level structure ... jet of trivialization (if ∇ has pole of order > 1)

$\exists$ notion of Hecke eigensheaf for a $\mathcal{D}$-module on $\mathrm{Bun}_{G,?}$ with eigenvalue $E|_{X - \{y_1 \dots y_n\}}$

Q : 1. How to construct objects of $\mathcal{A}ut_E$

2. How to describe the category $\mathcal{A}ut_E$ in dual terms

Example of an answer to 2: $\mathcal{A}ut_E \simeq \mathrm{Vect}$
... when $\exists!$ irreducible object & all other objects are direct sums.

Another kind of answer: $D^b(\mathcal{A}ut_E) \simeq D^b(\mathcal{O}_{Sp_N}\text{-mod})$

$\mathcal{O}$-modules on Springer fiber over N = unipotent monodromy of ∇ around y, or a dg version thereof.

$(N \in {}^LG)$

Physics picture : T-duality

$$\begin{tikzcd}
\mathcal{L}oc_{{}^LG} \simeq \mathcal{M}_{H,{}^LG} \arrow[dr] & & \mathcal{M}_{H,G} \arrow[dl] \arrow[r, squiggly] & \mathcal{D}\text{-modules on } Bun_G \\
& H & &
\end{tikzcd}$$

$E \in \mathcal{L}oc_{{}^LG}$

Important lesson from physics: complexity of category $\mathcal{A}ut_E$ related to singularity of $\mathcal{M}_{H,{}^LG}$ at E!!

Q1 in the unramified case:

$$\mathrm{Bun}_G \simeq G_{out} \backslash G(K_x)/G(\mathcal{O}_x)$$

for any $x \in X$: $K_x = \mathbb{C}((t_x))$

$\mathcal{O}_x = \mathbb{C}[[t_x]]$

(for G semisimple)

$G_{out} = G(\mathbb{C}[X \setminus x])$

Beilinson-Bernstein:

construct a functor $\Delta : (\mathfrak{g}, K)\text{-mod} \longrightarrow \mathcal{D}_{H\backslash G/K}\text{-mod}$

whenever we have a double coset

$H \backslash G/K$.

$(\mathfrak{g}, K)$-module: $\mathfrak{g}$-module V

which integrates to $K \subset G$.

Let's denote $(\mathfrak{g}, K)$-mod by $\mathfrak{g}\text{-mod}^K$.

e.g. case $K = \{1\}$ $\mathfrak{g}\text{-mod} \longrightarrow \mathcal{D}_{H\backslash G}\text{-mod}$

$$M \longmapsto \Delta(M) = \mathcal{D}_{H\backslash G} \otimes_{U\mathfrak{g}} M$$

using action of $U\mathfrak{g} \longrightarrow \Gamma(\mathcal{D}_{H\backslash G})$

Refs math.QA/0512172
(reviews math.QA/0611294
by E.F.)

Now we'll take for $\mathfrak{g}$ the loop algebra
$\mathfrak{g}(K_x) = \mathfrak{g}((t_x))$, $K \rightsquigarrow G(\mathcal{O}_x)$,
$H \rightsquigarrow G_{out}$

Δ. $\mathfrak{g}(K_x)\text{-mod}^{G(\mathcal{O}_x)} \longrightarrow \mathcal{D}_{Bun_G}\text{-mod}$

now use instead the affine Kac-Moody algebra
$\hat{\mathfrak{g}}_x$, (universal) central extension of $\mathfrak{g}(K_x)$

Δ: $\hat{\mathfrak{g}}\text{-mod}^{G(\mathcal{O}_x)} \longrightarrow \mathcal{D}^{tw}_{Bun_G}\text{-mod}$

[use central extension variant of B-B :
$1 \to \mathbb{C}^* \to \hat{G} \to G \to 1 \Rightarrow$ line bundle

$\mathcal{L}$: $H \backslash \widehat{G}/K$. $\hat{\mathfrak{g}}\text{-mod} \longrightarrow \mathcal{D}$-modules twisted
$\downarrow \mathbb{C}^*$ by $\mathcal{L}$.
$H \backslash G/K$]

Particular central extension: the <u>critical level</u>.

$$\begin{tikzcd} \Delta: \hat{\mathfrak{g}}_{crit}\text{-mod}^{G(\mathcal{O}_x)} \arrow[r] \arrow[dr] & \mathcal{D}_{Bun_G, K^{\frac{1}{2}}}\text{-mod} \arrow[d, "\wr"] & F \arrow[d, mapsto] \\ & \mathcal{D}_{Bun_G}\text{-mod} & F \otimes K^{-\frac{1}{2}} \end{tikzcd}$$

When we apply Δ to the right objects, get Hecke eigensheaves!

Key point: $\hat{\mathfrak{g}}_{crit}$-mod fibers over the space $Op_{{}^LG}(D_x^*)$ .... i.e. is <u>linear</u> over the algebra of functions $\mathrm{Fun}\, Op_{{}^LG}(D^*)$ on the affine space of <u>${}^LG$-opers</u> (this algebra acts on all objects in $\hat{\mathfrak{g}}_{crit}$).

— follows from theorem of Feigin–Frenkel identifying this ring of functions with the center of $U\hat{\mathfrak{g}}_{crit}$.

affine space

$$\mathrm{Op}_{{}^LG}(X) = \{(\mathcal{F}, \nabla, \mathcal{F}_{{}^LB})\}$$ reduction to ${}^LB$ satisfying strong transversality condition

$$\downarrow$$

$$\mathrm{Loc}_{{}^LG}(X) = \{(\mathcal{F}, \nabla)\}$$ complicated stack

$\boxed{\text{NB:}}$ the local systems coming from opers for X compact are always <u>irreducible</u>.

Moreover in this case the oper structure $\mathcal{F}_{{}^LB}$ is in fact <u>unique</u>.

<u>In fact</u> the underlying ${}^LG$ bundle $\mathcal{F}$ is <u>unique</u> (up to choice of θ-characteristic).

& all connections on this $\mathcal{F}$ are oper connections

So $\mathrm{Op}_{{}^LG}(X) =$ Connections of fixed $\mathcal{F}_0$ for X projective.

Construction of Hecke eigensheaves:

$$\chi \in \mathrm{Op}_{{}^LG}(X) \Rightarrow \chi_x = \chi|_{D_x^*} \in \mathrm{Op}_{{}^LG}(D_x^*)$$

Fact: $\exists!\ \mathbb{V}_\chi \in \hat{\mathfrak{g}}_{x,\mathrm{crit}}\text{-mod}^{G(\mathcal{O}_x)}$ (irreducible) on which the center acts according to $\chi_x \in Op_{{}^LG}(D_x^*)$

Theorem (Beilinson–Drinfeld) $\Delta(\mathbb{V}_{\chi,x})$ is a Hecke eigensheaf on Bun_G with eigenvalue $E = E_\chi$.

Expectation: generically Δ is an equivalence in some sense.

Ramified situation $y_1, \dots, y_n \in X$ ramification points.

$$Bun_{G,(y_i),par} \simeq G_{out} \backslash \prod G(K_{y_i}) / \prod_i I_{y_i}$$

I_{y_i} = Iwahori subgroup $\subset G(\mathcal{O}_{y_i})$ (takes value in B at y_i)

$$\Delta : \bigotimes \hat{\mathfrak{g}}_{y_i, crit}\text{-mod}^{I_{y_i}} \longrightarrow \mathcal{D}_{Bun_{G}, (y_i), par}\text{-mod}$$

(assume only one $y = y_1$ for simplicity).

$E = (\mathcal{F}, \nabla)$

suppose $E = E_\chi$ for χ an oper on $X \setminus y_1, \dots, y_n$ with regular singularities at $y_1, \dots, y_n$. (connection & ${}^L B$ reduction satisfy transversality outside the points y_i)

Restrict the oper χ near y.

$\chi|_{D_x^*} \in Op_{{}^L G}(D_x^*) \Rightarrow$ get category

<u>Def</u> $\hat{\mathfrak{g}}_{y, crit}\text{-mod}_{\chi_y}$ = category of $\hat{\mathfrak{g}}_{y, crit}$ modules, on which Fun $Op_{{}^L G}(D^*)$ acts by evaluation at χ_y

$$\Rightarrow \Delta : \hat{\mathfrak{g}}_{y, crit}\text{-mod}_{\chi_y} \longrightarrow \mathcal{D}_{Bun_{G}, (y), par}\text{-mod}$$

Beilinson-Drinfeld : for any M, $\Delta(M)$ is a Hecke eigensheaf with eigenvalue $E = E_\chi$.

i.e the functor takes $\hat{\mathfrak{g}}_{y,\mathrm{crit}}\text{-mod}^{I_y}_{\chi_y}$ to our Hecke eigencategory Aut_{E_x}.

[Unramified case: $\hat{\mathfrak{g}}\text{-mod}^{G(O)}_{\chi_x} \simeq \mathrm{Vect}$

$$\begin{array}{ccc} \mathbb{V}_x & \longleftrightarrow & \mathbb{C} \end{array}$$]

In the ramified case the local category is already a lot more complicated, so expect categories of Hecke eigensheaves to be much more complicated.

Gaitsgory - Frenkel : describe these local categories & get statements about Hecke eigensheaves.