

# Edward Frenkel - Ramified Geometric Langlands III

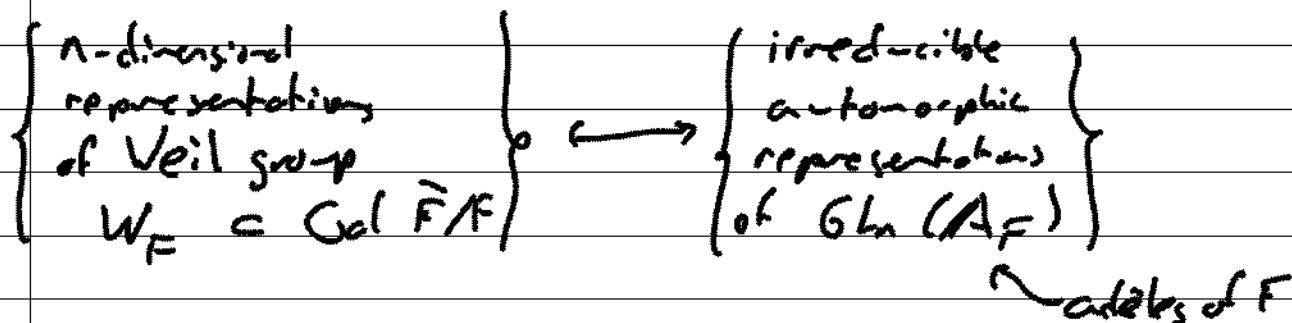
Note Title

1/18/2007

## Classical Langlands Correspondence for Function Fields

$X$  smooth projective curve /  $\mathbb{F}_q$

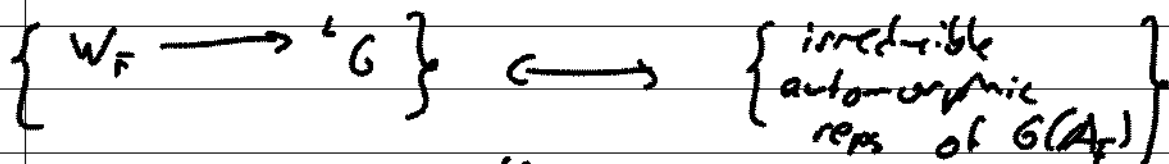
$F = \mathbb{F}_q(X)$  Galois group of separable closure



Automorphic: appears in functions

on  $\text{GL}_n F \backslash \text{GL}_n \mathbb{A}_F \dots$  To get precise statement need to modify to deal with continuous spectrum .. → Theorem of Lafforgue

(Drinfeld for  $\text{GL}_2$ )



↕ geometrize/category

$\mathcal{F}, \mathcal{V}$   ${}^L G$  local system on  $X/\mathbb{C}$   $\longleftrightarrow$   $\mathcal{A}, \mathcal{E}$  category of Hecke eigenstates

Question What eigenvalues on what?

$$G(A) = \prod'_{x \in X} G(F_x) \quad (g_x \in G(\mathcal{O}_x) \text{ all but fin. many } x)$$

Any irred. rep  $\pi \simeq \bigotimes'_{x \in X} \pi_x$  local factors

$\pi_x$  is an irrep of  $G(F_x)$  all  $x$

For all but fin many  $x$ ,  $\pi_x^{G(\mathcal{O}_x)} \neq 0$  (hence 1-dim).

Choose  $0 \neq v_x \in \pi_x^{G(\mathcal{O}_x)}$  at all these  $x$

$$\bigotimes' \pi_x = \left\{ \bigotimes'_x u_x : u_x \in \pi_x, u_x = v_x \text{ for all but fin many } x \right\}$$

Galois side:  $\text{Gal}(F_x/F_x) \subset \text{Gal}(\bar{F}/F)$  up to conjugation

Local Langlands correspondence:

$$\left\{ \sigma_x : \text{Gal } \bar{F}_x/F_x \rightarrow {}^L G \right\} \begin{matrix} \text{unrealized except at fin many pts} \\ \text{---} \end{matrix} \left\{ \begin{matrix} \text{irreducible smooth} \\ \text{representations of} \\ G(F_x) \end{matrix} \right\}$$

GL<sub>n</sub>: Theorem by Langlands - Shimura - Rapoport.

$$\sigma_x \longmapsto \pi_x(\sigma_x)$$

Compatibility of local & global correspondences:

$$\begin{array}{ccc} \sigma & \longmapsto & \Pi(\sigma) = \bigotimes' \Pi_x(\sigma_x) \\ \downarrow & & \uparrow \\ \sigma_x = \sigma|_{G_{\text{ad}} \bar{F}_x/F_x} & \longmapsto & \Pi_x(\sigma_x) \end{array}$$

and  $\Pi(\sigma)$  realized in  $\text{Fun}(G(F) \backslash G(\mathbb{A}))$

Let's categorify  $\Pi_x(\sigma_x) \rightsquigarrow$  category  $\mathcal{C}_{\sigma_x}$   
with  $G(\mathbb{Q}((t)))$  formal loop group

[Ref: F., Langlands Correspondence for Loop Groups]

Very big... let's first categorify fin dim  
subspaces of  $\Pi_x$ .

$K_x \subset G(\mathcal{O}_x)$  compact group,

e.g.  $G(\mathcal{O}_x)$  or  $I_x$  Iwahori. or  $P_x$  parabolic  
or congruence subgroup

$\Rightarrow$  can replace  $\Pi_x$  by  $\Pi_x^{K_x}$

( $\Pi_x$  smooth  $\rightsquigarrow$  fin dim vector space)

$$\pi_x^{K_x} \circlearrowleft H(G(F_x), K_x) = \left( C_c(G(F_x))^{K_x \times K_x}, * \right)$$

Hecke algebra compactly supported

eg  $H(G(F_x), G(O_x)) \simeq \text{Rep}^c G$

$H(G(F_x), I_x)$  affine Hecke algebra with parameter  $q = |F_x|$ .

$\sigma_x$  unramified  $\Rightarrow (\pi_x(\sigma_x))^{G(O_x)} \neq 0$

Choose  $K$  compact subgroup of  $G(A_x)$ ,  $K = \prod_x K_x$   
 $K_x = G(O_x)$  at all but fin many

Ramification at points  $y_i$  forces us to choose level structures: compact subgroups  $K_{y_i} \subset G(O_{y_i})$  so that  $(\pi_{y_i})^{K_{y_i}} \neq 0$ .

Two competing tendencies:

1. want  $K_{y_i}$  small so that  $(\pi_{y_i})^{K_{y_i}} \neq 0$
2. want  $(\pi_{y_i})^{K_{y_i}}$  as small as possible ---- ideally one-dimensional, so want  $K_{y_i}$  big.

Question: is it always possible to choose  $K_{y_i}$   
 so that  $(\prod_{y_i} K_{y_i})$  is one-dimensional?

---- insight from Gukov-Witten ...

Take  $\Pi(\sigma)^K$  finite dim vector space  
 with  $\otimes H(G(F_x), K_x)$  action

As vector space  $\Pi(\sigma)^K \simeq \otimes_{y_i \text{ ramification points}} \prod_{y_i} K_{y_i} \otimes \text{Hecke}$

Automorphy:  $\Pi(\sigma)$  realized in  $\text{Fns}(G(F) \backslash G(A))$

$\Pi(\sigma)^K$  realized in  $\text{Fns}(G(F) \backslash G(A)/K)$

=  $G$ -bundles with  $K_{y_i}$  level structure at  $y_i$ .

Geometric world  $X/\mathbb{C}$

$E = (F, D)$   $\hookrightarrow$  connection on  $X$  with poles at  $y_1, \dots, y_n$

$\rightsquigarrow \text{Aut}_E$  category of  $D$ -modules on

$\text{Bun}_G(y_i, K_{y_i})$  with Hecke eigenstate property

at  $x \neq y_i$  with eigenvalue  $E_x$

& action of ramified Hecke algebras at  $y_i$ :

action of  $D\text{-mod}(\mathcal{X}_n \setminus G(F_n)/K_n)$ .

(an action of affine braid group where  $K_n = \mathbb{I}_n$   
Lusztig: level structure).

Physics point of view: ability to find  
single eigenstate relates to existence  
of a single B-brane at a point

$\longleftrightarrow$  singularity of this point on the Hitchin  
moduli space.

Metaconjecture Can choose  $K_{Y_i}$  so that  
 $\mathbb{P}_{Y_i}^{K_{Y_i}}$  is 1-dimensional whenever the  
corresponding point of the Hitchin  
moduli space is smooth