Edward Frenkel - Ramified Geometric Langlands III

Note Title 1/18/2007

Classical Langlands Correspondence for Function Fields

X smooth projective curve $/\mathbb{F}_q$

$F = \mathbb{F}_q(X)$. $\mathrm{Gal}(\bar{F}/F)$ Galois group of separable closure

$$\left\{\begin{array}{l}\text{n-dimensional} \\ \text{representations} \\ \text{of Weil group} \\ W_F \subset \mathrm{Gal}\,\bar{F}/F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{irreducible} \\ \text{automorphic} \\ \text{representations} \\ \text{of } GL_n(\mathbb{A}_F)\end{array}\right\}$$

adeles of F

Automorphic: appears in functions on $GL_n F \backslash GL_n \mathbb{A}_F$ ... To get precise statement need to modify to deal with continuous spectrum .. $\to$ Theorem of Lafforgue (Drinfeld for GL_2)

$$\left\{ W_F \longrightarrow {}^L G \right\} \longleftrightarrow \left\{\begin{array}{l}\text{irreducible} \\ \text{automorphic} \\ \text{reps of } G(\mathbb{A}_F)\end{array}\right\}$$

$\Downarrow$ geometrize/categorify

$\mathcal{F}, \nabla$ ${}^L G$ local system on $X/\mathbb{C}$ $\longleftrightarrow$ Aut_E category of Hecke eigensheaves

<u>Question</u> Hecke eigensheaves on what?

$G(\mathbb{A}) = \prod'_{x \in X} G(F_x)$ $\left(g_x \in G(\mathcal{O}_x) \text{ all but fin. many } x \right)$

Any irred. rep $\pi \simeq \bigotimes'_{x \in X} \pi_x$ local factors

π_x is an irrep of $G(F_x)$ all x

For all but fin many x, $\pi_x^{G(\mathcal{O}_x)} \neq 0$ (hence 1-dim).
Choose $0 \neq v_x \in \pi_x^{G(\mathcal{O})}$ at all these x

$\bigotimes' \pi_x := \{ \bigotimes_x u_x : u_x \in \pi_x, \ u_x = v_x \text{ for all but fin many } x \}$

Galois side: $Gal(\overline{F}_x/F_x) \subset Gal(\overline{F}/F)$ up to conjugation

Local Langlands correspondence:

$$\left\{ \sigma_x : Gal\, \overline{F}_x/F_x \longrightarrow {}^L G \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible smooth} \\ \text{representations of} \\ G(F_x) \end{array} \right\}$$

unramified except at fin many pts

GL_n: Theorem by Laumon - Stuhler - Rapoport.

$$\sigma_x \longmapsto \pi_x(\sigma_x)$$

Compatibility of local & global correspondences:

$$\begin{tikzcd}
\sigma \arrow[r, mapsto] \arrow[d] & \pi(\sigma) = \bigotimes' \pi_x(\sigma_x) \\
\sigma_x = \sigma|_{\mathrm{Gal}\,\overline{F}_x/F_x} \arrow[r, mapsto] & \pi_x(\sigma_x) \arrow[u]
\end{tikzcd}$$

and $\pi(\sigma)$ realized in $\mathrm{Fun}(G(F)\backslash G(\mathbb{A}))$

Let's categorify $\pi_x(\sigma_x) \rightsquigarrow$ category $\mathcal{C}_{\sigma_x}$
with $G(\mathbb{C}((t)))$ formal loop group

[Ref: F., Langlands Correspondence for Loop Groups]

Very big.... let's first categorify fin dim subspaces of π_x.

$K_x \subset G(\mathcal{O}_x)$ compact group,
e.g. $G(\mathcal{O}_x)$ or I_x Iwahori. or P_x parahori
or congruence subgroup

$\Rightarrow$ can replace π_x by $\pi_x^{K_x}$

(π_x smooth $\rightsquigarrow$ fin-dim vector space)

$\Pi_x^{K_x} \circlearrowleft H(G(F_x), K_x) = \left(C_c\left(G(F_x)\right)^{K_x \times K_x}, *\right)$

Hecke algebra compactly supported

eg $H(G(F_x), G(O_x)) \simeq \mathrm{Rep}\ {}^L G$

$H(G(F_x), I_x)$ affine Hecke algebra with parameter $q = |\mathbb{F}_q|$.

σ_x unramified $\Rightarrow (\Pi_x(\sigma_x))^{G(O_x)} \neq 0$

Choose K compact subgroup of $G(\mathbb{A}_x)$, $K = \prod_x K_x$

$K_x = G(O_x)$ at all but fin many

Ramification at points y_i forces us to choose level structures: compact subgroups $K_{y_i} \subset G(O_{y_i})$ so that $(\Pi_{y_i})^{K_{y_i}} \neq 0$.

Two competing tendencies:

1. want K_{y_i} small so that $(\Pi_{y_i})^{K_{y_i}} \neq 0$

2. want $(\Pi_{y_i})^{K_{y_i}}$ as small as possible --- ideally one-dimensional, so want K_{y_i} <u>big</u>.

Question: is it always possible to choose K_{y_i} so that $(\Pi_{y_i})^{K_{y_i}}$ is one-dimensional?

---- insight from Gaitsgory-Witten ...

Take $\pi(\sigma)^K$ finite dim vector space with $\bigotimes H(G(F_x), K_x)$ action

As vector space $\pi(\sigma)^K \simeq \bigotimes_{\substack{y_i \text{ ramification} \\ \text{points}}} \Pi_{y_i}^{K_{y_i}} \circlearrowleft H_{\text{Hecke}}$

Automorphy: $\pi(\sigma)$ realized in $\text{Fns}(G(F)\backslash G(\mathbb{A}))$

$\pi(\sigma)^K$ realized in $\text{Fns}(G(F)\backslash G(\mathbb{A})/K)$

$=$ G-bundles with K_{y_i} level structure at y_i.

Geometric world $X/\mathbb{C}$

$E = (\mathcal{F}, \nabla)$ ${}^L G$ connection on X with poles at $y_1, \dots y_n$

$\rightsquigarrow \mathcal{A}ut_E$ category of $\mathcal{D}$-modules on $\text{Bun}_{G,(y_i),(K_{y_i})}$ with Hecke eigensheaf property at $x \neq y_i$ with eigenvalue E_x

& action of ramified Hecke algebras at y_i:

action of $\mathcal{D}\text{-mod}\left(K_x \backslash G(F_x) / K_x\right)$.

(~ action of affine braid group where $K_x = I_x$ Iwahori level structure).

Physics point of view: ability to find <u>single</u> eigenbrane relates to existence of a <u>single</u> B-brane at a point

$\longleftrightarrow$ singularity of this point on the Hitchin moduli space.

<u>Meta conjecture</u> Can choose K_{y_i} so that $\Pi_{y_i}^{K_{y_i}}$ is 1-dimensional whenever the corresponding point of the Hitchin moduli space is smooth