Sergei Gukov – Gauge theory & tamely ramified Langlands

Note Title 1/15/2007

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4d topological gauge theory on $M = C \times \Sigma$

$\xrightarrow[C \text{ small}]{\sim}$ 2d topological σ-model on $\mathcal{M}_H(G, C)$

Ramified case: allow singularities on C.

S-duality:
$$G \longleftrightarrow {}^L G$$
$$\tau \longleftrightarrow -\frac{1}{n_g \tau}$$

A-model on $\mathcal{M}_H(G, \text{ramification}) \longleftrightarrow$ B-model on $\mathcal{M}_H({}^L G, {}^L\text{ramification})$

A-branes $\longleftrightarrow$ B-branes

't Hooft loops $\longleftrightarrow$ Wilson loops

* OPE algebra of line operators becomes noncommutative in the ramified case.

Outline:
1. local model
2. surface operators and duality
3. "global version"

Local Model Combined system: $M = C \times \Sigma$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \ni p$

4d gauge theory on M coupled to
2d topological σ-model on $p \times \Sigma$
$$\Sigma \to Q$$

To couple this to bulk gauge theory, require that Q is hyperkähler & carries a G-action
→ to preserve SUSY

- $\mathbb{L} \subset G$ Levi subgroup

Example: $\mathbb{L} = \mathbb{T} \subset G$ maximal torus.
In this case take Q: complex coadjoint orbit $G_{\mathbb{C}} / \mathbb{L}_{\mathbb{C}} \longleftrightarrow T^*(G/\mathbb{L})$
(in different complex structures), admit hyperkähler metric

Toy model: $G = SU_2 \quad \mathbb{L} = U(1)$
hyperkähler quotient $\quad Q = \mathbb{H}^2 /\!/\!/ U(1)$

Moment map $\vec{\mu} = (\mu_I, \mu_J, \mu_K) = (\alpha, \beta, \gamma)$ ← real moment map

Model	Complex modulus	Kähler modulus
I	$\beta + i\gamma$	α
J	$\gamma + i\alpha$	β
K	$\alpha + i\beta$	γ

$SO(3)$ rotates α, β, γ

$\alpha = \beta = \gamma = 0 \implies$ get A_1 singularity

$Q \rightsquigarrow \mathbb{R}^4/\mathbb{Z}_2$

Turning on $\alpha \neq 0 \implies$ get resolution

$Q \longrightarrow T^*\mathbb{CP}^1$ Springer resolution of nilpotent cone

In Quantum Theory: have another parameter

$\eta \longleftrightarrow$ B-field through the exceptional divisor $\mathbb{P}^1$

$\implies$ gives complexified Kähler parameter

eg instanton action would have weight

$\int_{\mathbb{P}^1} (B + i\omega)$ (eg η + iα in I structure

Symmetry in theory ("shifting τ"): $\eta \mapsto \eta + 1$.

So parameter space for topological σ-model is

$\mathbb{R}^3 \times S^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ negates $\alpha\ \beta\ \gamma$ & η (Weyl group symmetry).

Fixed points: P_0 : $\alpha = \beta = \gamma = \eta = 0$

At P_0 get a singular QFT

P_1 : $\alpha = \beta = \gamma = 0, \quad \eta = \frac{1}{2}$: (orbifold) Gepner point
.... get nonsingular QFT

At P_0, P_1 get symmetry w of the theory.

<u>B-model</u> on Q In complex structure J

* symmetries of the B-model $D^b(Q)$:
use derived McKay correspondence $D^b(Q) \simeq D^b_{\mathbb{Z}_2}(\mathbb{C}^2)$
(set $\gamma + i\alpha$ & vary $\beta + i\eta$)

The category $D^b_{\mathbb{Z}_2}(\mathbb{C}^2)$ has two objects

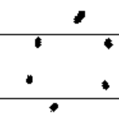
$$S_i = \rho_i \otimes \mathcal{O}_p \qquad \rho_i = \text{irrep of } \Gamma = \mathbb{Z}_2$$
$$\mathcal{O}_p = \text{skyscraper at origin}$$

.... "fractional branes"

S_i are spherical objects:

$$\mathrm{Ext}^i(S, S) = \begin{cases} \mathbb{C} & i = 0 \text{ and } d \\ 0 & \text{otherwise} \end{cases}$$

$\longrightarrow$ expect to correspond to Lagrangian spheres in mirror

A_n chain of spherical objects 

$$\sum_k \dim \mathrm{Ext}^k(S_i, S_j) = \begin{cases} 1 & |i-j|=1 \\ 0 & |i-j|>1 \end{cases}$$

Seidel-Thomas: The corresponding twist functors T_{S_i} give action of the affine braid group $B_{\mathrm{aff},n}$ on $D^b(Q)$

(here $n=2$)

W_{aff}: For simply laced G of rank >1

... generated by elementary reflections T_i

$T_i T_j T_i = T_j T_i T_j \qquad i,j = 1, \dots, r+1$

when the vertices are adjacent, and $T_i T_j = T_j T_i$ otherwise

+ relation $T_i^2 = 1$ (drop this $\Rightarrow B_{\mathrm{aff}}$).

* in our case: identify B_{aff} as symmetries of the Bridgeland

$$\mathrm{Autoeq}(D^b(Q)) = \underset{\text{shift}}{\mathbb{Z}} \cdot \mathrm{Aut}\,\Gamma \ltimes B_{\mathrm{aff}}(\Gamma)$$

Γ = affine Dynkin diagram [Bridgeland]

$\Gamma = \mathbb{Z}/2$: Aut $\Gamma = \mathbb{Z}_2$ generated by R

B_{aff} : $\langle A, B \rangle$ with $AR = RB$.

Since $B = RAR$, the group we're considering is generated by A & R with a single relation $R^2 = 1$.