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Note Title 1/17/2007

Last time: group of symmetries on B-branes on $\mathbb{R}^4/\mathbb{Z}/2$ is $\widehat{B}_{aff} = B_{aff} \ltimes Z(G)$

... generated by A, R $\quad R^2 = 1$

($\mathbb{R}^4/\Gamma$ ADE find $\widehat{B}_{aff} = B_{aff} \ltimes Z(G)$ acting)

$W_{aff} = \Lambda_{coroot} \ltimes W$

— eg $G = SU(2)$, $\Lambda = \mathbb{Z}$, $W = \mathbb{Z}/2$ gen by $A: n \longmapsto -n$,

W_{aff} is generated by $A, T: n \longmapsto n+1$

$A^2 = 1$, $ATA = T^{-1}$

equivalently have presentation $W_{aff} = \langle A, B \rangle$

$B = AT$: $A^2 = 1 = B^2$

To get B_{aff} drop these two relations

The braid group and monodromies

Q = resolved A_1 singularity

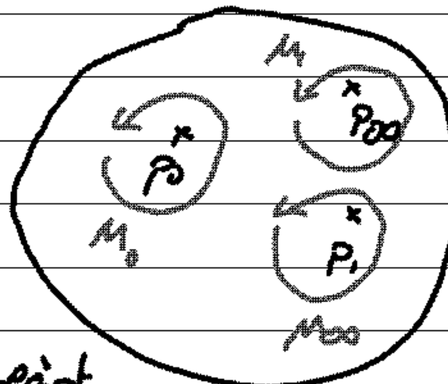
$D^b(Q)$ (B-branes) depends on complex structure parameter $Y = \exp(2\pi(\gamma + i\alpha))$

& is independent of $\mathbb{C}$-Kähler structure $\lambda = \exp(2\pi(\beta + i\eta))$

Now keep Y fixed but vary λ
& find monodromies in λ even though category
of B-branes is infinitesimally independent
of $\mathbb{C}$-Kähler parameter.

Kähler moduli space:

P_0 : $\lambda=1$ conifold point —
singular QFT



P_1 : $\lambda=-1$ orbifold/Gepner point
– solvable CFT

$M_1 = M_0 M_\infty$

P_∞ : $\lambda=\infty$ "large volume limit"
(γ = Kähler modulus $\gamma \to \infty$ curvature $\to 0$
B-field η valued in S^1, gives parameter
on loop around ∞)

Monodromies: study variation of branes around
loops M_0, M_1, M_∞.

M_0: around here have no η, everything is geometric,
so we're just collapsing a cycle $\Longrightarrow$

Monodromy M_0 = twist functor on $D^b(Q)$
associated with the spherical object
$\mathcal{O}_E$, E = exceptional divisor $\simeq \mathbb{P}^1$

M_1 : monodromy of order 2 $(M_1^2 = 1)$
Together M_0, M_1 generate $\mathrm{Autoeq}\, D^b(Q)$

we have field theories + sets of branes
depending on λ : 0-brane on point,
D2 brane on exceptional divisor &
Space filling D4 brane
... situation formalized by Douglas et al.
& Bridgeland. No infinitesimal dependence
on λ but do get global effects.

P_1 : solvable CFT : Geometry is $\mathbb{R}^4/\mathbb{Z}_2$
with half integrable period of B-field,
get automorphism $(\mathbb{Z}/2)$ of the category
of branes, which gives $\mathbb{Z}/2$ local monodromy.

.... have orbifold $\mathbb{R}^4/\mathbb{Z}/2$ with
frozen half-integral flux ... non geometric
(quantum) data, even though geometry

is conifold geometry the quantum effects dramatically — get an orbifold point of the moduli space, local orbifold π_1 is $\mathbb{Z}/2$.

D-brane charges take values in K-theory of Q.

On level of charges $M_0 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$

$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$ $\quad M_\infty = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$

In basis of D0, D2, D4 brane

$v(\mathcal{E}) = e^{iB} ch(\mathcal{E}) \sqrt{Td(Q)}$ Mukai vector — charge in cohomology

monodromy: $\eta = B \longmapsto B + 2\pi$

.... check $M_0^2 = 1$, $M_1^2 = 1$.

[Here B-field is flat — no H-flux/gerbes.]

The affine braid group is the fundamental group (group of monodromies) of the Kähler moduli space parametrized by $\lambda = \exp(2\pi(\beta + i\eta))$

Case 1: $Y = \exp(2\pi(\gamma - i\alpha)) = 1$ (fixed cx structure)

Recall $(\beta + i\eta) \in \mathbb{R} \times S^1 / \mathbb{Z}/2$:

fixed pts $\lambda = \pm 1$

Monodromies: A = monodromy around $\lambda = 1$

R = monodromy around $\lambda = -1$: orbifold monodromy

Case 2: $Y \neq 1$: the space of parameters (β, η) is $\mathbb{R} \times S^1$, no residual $\mathbb{Z}/2$ action ... don't hit orbifold singularity once complex moduli turned on.

$\Rightarrow$ monodromy group is $\mathbb{Z}$

The group acting on B-branes is

$\widehat{B_{aff}} = B_{aff} \ltimes Z({}^LG)$ if $Y = 1$ (singular)

$\Lambda_{cochar} = \Lambda_{cort} \ltimes Z({}^LG)$ if Y is regular

... in higher rank have many intermediate choices !

$\boxed{\text{Gauge theory}}$: above simple setup was local, 2D TFT which has no S-dual! we want to couple it to gauge theory.

<u>Locally</u> M 4-manifold $= \underset{\Sigma\text{-direction}}{D} \times \underset{C\text{-direction}}{D'}$

D': section of normal bundle to surface singularity. with coord $z = x^1 + i x^2 = re^{i\theta}$

Hitchin's equations: $\begin{cases} F - \phi \wedge \phi = 0 \\ d_A \phi = 0 \qquad d_A * \phi = 0 \end{cases}$

Rotation invariant pair (A, ϕ) : local ansatz

$A = a(r)\, d\theta + h(r) \frac{dr}{r}$

$\phi = b(r) \frac{dr}{r} - c(r)\, d\theta$

such that at $r=0$: $\begin{cases} A = \alpha\, d\theta + \cdots \\ \phi = \beta \frac{dr}{r} - \gamma\, d\theta + \cdots \end{cases}$

a, b, c, h $\mathfrak{g}$-valued functions of r

$\alpha, \beta, \gamma \in \mathfrak{t}$.

If we set $s = -\ln r$ $h = 0$ (choice of gauge)

$\Rightarrow$ Hitchin equations reduce to Nahm's eqns

$SU(2)$: $\frac{da}{ds} = [b, c]$ $\frac{db}{ds} = [c, a]$ $\frac{dc}{ds} = [a, b]$

Example: $\alpha = \beta = \gamma = 0$

get a trivial solution. $a = b = c = 0$

& a nontrivial solution $a = -\frac{1}{s} t_1$ $b = -\frac{1}{s} t_2$ $c = -\frac{1}{s} t_3$

where t_1, t_2, t_3 are $\mathfrak{su}_2$ triple

$[t_i, t_j] = t_k$ .... come in 4-parameter family

More generally: $R \in SO(3)$

$a = -\frac{1}{s + f^{-1}} R t_1 R^{-1}$ $b = -\frac{1}{s + f^{-1}} R t_2 R^{-1}$

$c = -\frac{1}{s + f^{-1}} R t_3 R^{-1}$

$\Rightarrow$ space of solutions in this local setting is given by $\underset{\substack{\shortparallel \\ f}}{\mathbb{R}_{\geq 0}} \times \underset{\text{angles}}{SO(3)} = \mathbb{R}^4/\mathbb{Z}_2$

.... recover our model space from before.

Turning on (α, β, γ) can still solve eqns $\Rightarrow$ get Eguchi-Hanson metric on resolved A_1 singularity $= Q$.