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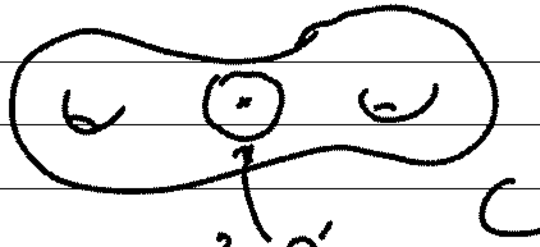
Note Title 1/18/2007

Surface Operators



$\mathbb{R}^2 \simeq D'$, coords $x^1 + ix^2 = re^{i\theta} = z$

Hitchin's equations:

$$\begin{cases} F - \phi \wedge \phi = 0 \\ d_A \phi = 0 \quad d_A * \phi = 0 \end{cases}$$

D: locus of the singularity $\subset C \times \Sigma$

Isolated singularity at the origin of D' ($\subset D' \times D$)

model by $\begin{cases} A = \alpha\, d\theta + \cdots \\ \phi = \beta \frac{dr}{r} - \gamma d\theta + \cdots \end{cases}$ $\quad \alpha, \beta, \gamma \in \mathfrak{t}$

... similar to definition of 't Hooft operator for line operators...

* Weyl group W of G acts on (α, β, γ)
* $\alpha \longmapsto \alpha + u \quad u \in \Lambda_{cochar}$, $\exp(2\pi u) = 1$

symmetries of the surface operator:

large gauge transformation by $(r, \theta) \longrightarrow \exp(\theta u)$ for u integral as above shifts α by u.

$\Rightarrow$ really $\alpha \in \mathfrak{t} / \Lambda_{cochar}$

$(\alpha, \beta, \gamma) \in \mathfrak{t} \times \mathfrak{t} \times \mathfrak{t} / \Lambda_{\text{cochar}} \ltimes W = \mathbb{T} \times \mathfrak{t} \times \mathfrak{t} / W$

(SU_2: $\mathfrak{t} = \mathbb{R}$, this becomes our old picture)

<u>For general 4-manifold M</u>:

G-bundle over M, symmetry broken along surface $D \subset M$ $\Rightarrow$ $\mathbb{T}$-bundle over D

Shifts in α come from G gauge transformations outside D: no natural extension of G-bundle to D but do get $\mathbb{T}$ symmetry (for α regular) $\Longrightarrow$ allows for introduction of <u>θ-angles</u>:

$T = U(1)^r$; a $U(1)$ bundle $\mathcal{L}$ on D is quantized by integrality of

$d = \int_D c_1(\mathcal{L})$ $\Rightarrow$ can add new terms

of form $\exp(i\eta d)$ in action

For $\mathbb{T}$-bundle over D: classified by monopole charge $m \in \Lambda_{\text{cochar}} = \text{Hom}(U(1), \mathbb{T})$

$\Rightarrow$ $\eta : \Lambda_{\text{cochar}} \longrightarrow U(1)$ i.e. $\eta \in {}^L\mathbb{T} = \mathfrak{t}^* / \Lambda_{\text{char}}$

So $(\alpha, \beta, \gamma, \eta) \in \mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times {}^L\mathbb{T} / W$

α, β, γ are classical & η is quantum

(η corresponds to B-field on parabolic Hitchin space in 2d reduction, lives in H^2 of the space)

--- θ angle of 2d gauge theory on D

... in H^2 of Hitchin space get $1+r$ dim space, 1 universal θ-angle of 4d theory + r θ-angles for abelian theory at singularity,

Duality: $S:$ $G \longleftrightarrow {}^LG$

$\tau \longleftrightarrow {}^L\tau = -\frac{1}{n_g \tau}$

$(\beta, \gamma) \longleftrightarrow (\beta, \gamma)$

(for suitable choice of invariant form identifying $\mathfrak{t} \cong \mathfrak{t}^*$)

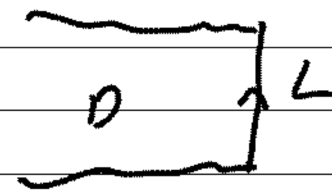
$(\alpha, \eta) \longleftrightarrow ({}^L\alpha, {}^L\eta) = (\eta, -\alpha)$

... generalization of S-duality conjecture

Evidence for interchange of α, η:

i) abelian case

ii) Boundary version: allow D to have a boundary line



... since have holonomy around D in 4-dim, must have electric/magnetic charge along L that will allow this holonomy to "terminate" along L

$\Rightarrow$ find L carries magnetic charge α & electric charge η

- similar to Dirac string: string terminating on monopole : D is worldsheet of Dirac string ...

$\Rightarrow$ expect α, η to be exchanged

MO duality (+ topological reduction):

A-model of $\mathcal{M}_H(\alpha,\beta,\gamma,\eta; p, G), \omega_K$ $\Leftrightarrow$ B-model of $\mathcal{M}_H(\eta,\beta,\gamma,-\alpha; p, {}^LG), J$

eg 0-brane in B-model of space of ramified local systems $\rightsquigarrow$ special A-brane (eigenbrane) on space of parabolic Higgs bundles.

't Hooft operators $\longleftrightarrow$ Wilson operators

Additional factor: Braid group $\mathcal{B}_\gamma$ or $\widehat{\mathcal{B}}_\gamma = \mathcal{B}_\gamma \ltimes Z(G)$

for $\gamma = \exp(-2\pi(\eta - i\gamma))$ in A-model

(Kähler modulus + B-field)

$$= \exp(2\pi({}^L\alpha + i\,{}^L\gamma)) \text{ in B-model}$$

(complex structure J depends on α, γ).

.... conj-gacy class of monodromy around G^{fin}

Have additional line operators stuck to surface operator D, more than in bulk.

$$\mathcal{B}_\gamma = \begin{cases} B_{aff} & \text{if } \gamma = 1 \\ \vdots & \text{intermediate parabolic versions} \\ \Lambda_{coroot} & \text{if } \gamma \text{ is regular} \end{cases}$$

B_y = (orbifold) fundamental group of parameters on which our model does not depend:
on A-side these are (α, β)
on B-side these are (η, β)
...... in this picture these operators act by monodromies

Much easier to compute these monodromies on A-side, where don't need the quantum parameter η, can calculate the braid group action geometrically.

Categorification: $W_{aff} = \Lambda_{coroot} \ltimes W$
(for G simply connected) is categorified by B: W_{aff} acts on
<u>charges</u> of branes, B_{aff} acts on branes themselves.

W_{aff} action on cohomology:
$$\pi_1\left(\frac{\mathfrak{t} \times \mathfrak{t} \times \mathfrak{t} \setminus \text{irregular}}{\Lambda_{coroot} \ltimes W}\right) = \Lambda_{coroot} \ltimes W$$
$$= W_{aff}$$
Singularities appear in codim three

Vary $\alpha, \beta, \gamma \Rightarrow W_{aff}$ action.

To categorify: only vary α, β (keep γ fixed)

but $\pi_1\left(\frac{\mathfrak{t}\times\mathfrak{t} - \text{sing}}{\Lambda \ltimes W}\right) = B_{aff}$

So categorification comes from $\mathfrak{t}\times\mathfrak{t}\times\mathfrak{t} \rightsquigarrow \mathfrak{t}\times\mathfrak{t}$
$(3 \rightsquigarrow 2)$

α, β, γ give parameters deforming principal nilpotent coadjoint orbit to regular semisimple ones, smoothing the orbit closure. The same picture extends to arbitrary Richardson orbits.