Anton Kapustin - TFT & Geometric Langlands

Note Title 1/16/2007

$M_4 = \Sigma \times C$ vol $C \ll$ vol Σ

$\rightsquigarrow$ limits to a field theory on Σ

(since we're in a topological theory the volume of C is unimportant ... get same picture for any volume ... but in more general QFT pick out certain states by this limit)

Bosonic fields in the theory: connection D & $\phi \in \Gamma(\Omega^1_{\Sigma \times C})$

Case $\Sigma = \mathbb{R}^2$: can show these must be pulled back from C.

Space of vacua: solns of self-duality // gauge

$$\begin{cases} F_C - \phi_C \wedge \phi_C = 0 \\ D_C \phi_C = 0 \\ D_C * \phi_C = 0 \end{cases} \Rightarrow \mathcal{M}_H(G, C)$$

Effective field theory is a σ-model with target $\mathcal{M}_H(G, C)$:

$\phi: \Sigma \longrightarrow \mathcal{M}_H(G, C)$ + fermionic fields:

no longer spinors as in 4d but certain forms

$\mathcal{M}_H(G,C)$ is hyperkähler & above eqns are moment map eqns for the gauge group action on space of (D_C, ϕ_C).

Distinguished complex structures

$$I: \quad \mathcal{M}_H(G,C)_I = \mathcal{M}_{Higgs}(G,C):$$

$$\bar{\partial}_A \phi_C^{1,0} = 0$$

$$J: \quad \mathcal{M}_H(G,C)_J = \mathcal{M}_{flat}(G_{\mathbb{C}}, C):$$

$$\mathcal{D}_C = D_C + i\phi_C \text{ is a flat } G_{\mathbb{C}} \text{ connection}$$

Topologically twisted σ-models with a HK target X are labelled by $\underset{w_+}{\mathbb{P}^1} \times \underset{w_-}{\mathbb{P}^1}$

$I_{w_+} = I_{w_-} \Rightarrow$ B-model with target X_w

$I_{w_+} = -I_{w_-} \Rightarrow$ A-model with target $(X, \omega_{I_{w_+}})$

$t = 1$: A-model of $(\mathcal{M}_H, \omega_K)$

$t = i$: B-model on $(\mathcal{M}_H, J)$

Matching of space of states on the circle:

On $\mathcal{M}_{flat}$ natural functions are

Wilson loops ... traces of monodromies.

On dual side get 't Hooft loops which are winding states: vertex operators on Σ

.... in this noncompact setting space of states is not usual de Rham cohomology = A-model states.

How to get B-model in complex structure I?

Relates to classical limit of geometric Langlands.

$$D^b(Coh(\mathcal{M}_{Higgs}(G,C))) \simeq D^b(Coh(\mathcal{M}_{Higgs}({}^LG,C)))$$

B-model in I on Σ $\longleftrightarrow$ B-model in I on Σ

— theory on $\Sigma \times C$ which is topological on Σ but only holomorphic on C

<u>Nontopological twists</u> : ∞ many possible twistings (preserving SUSY) on $\Sigma \times C$.

.... in fact 4 SUSYs survive on 4-manifolds of this form

$$Q_{BRST} = \sum_{1}^{4} u_i Q_i$$

Choice of BRST operator says what 2d TFT we'll get on $\mathcal{M}_H$. .... but now we have $\mathbb{P}^3$ worth of parameters & only wanted $\mathbb{P}^1 \times \mathbb{P}^1$.

For some choices of u_i end up with chiral algebras in 2d ... holomorphic 2d field theory on Σ. (never depends on Kähler forms on Σ, C only complex structure)

For <u>most</u> u_i get TFT on Σ.

Other parameters get half twisted model on $\mathcal{M}_{Higgs}$ — which gives chiral de Rham on microscopic strings

Theories with less SUSY:
$N=2$ $d=4$ theories sometimes have dualities similar to S-duality

Must restrict representations R in which fields live to have $c(R)=c(\mathrm{ad})$ same Casimir as adjoint

e.g. $G=SU_2$ $\quad R=\bigoplus_1^4 \mathbb{C}^2$
S-W theory: conjectured to have S-duality.

Can't be twisted to a TFT, but can twist into mixed TFT/CFT on $\Sigma \times C$:

$\Rightarrow$ B-model with target $\mathcal{M}_{\substack{\text{generalized}\\ \text{Higgs}}}(G,R,C)$

... instead of adjoint Higgs have Higgs in representation R

This theory is <u>self-dual</u> $\Rightarrow$ some subgroup of $SL_2\mathbb{Z}$ $(\Gamma_0(2))$ acts on $D^b\,\mathrm{Coh}(\mathcal{M}_{\mathrm{Higgs}}(G,R,C))$:

condition on Casimir guarantees the
target is Calabi-Yau.