Andrei Losev – Everything you wanted to know about QFT

Note Title 1/11/2007

Physicist dreams about QFT

1. Classical physics: laws of motion given by variational principle $\delta S(\varphi) = 0$ eqn
   φ_{cl} – solution to this equation

2. Feynman path integral as a dream of QFT: formulate deformation of classical physics variational equation into object

$$I = \int \mathcal{D}\varphi \, e^{\frac{i}{\hbar} S(\varphi)}$$

& correlators of local observables

$$\langle \varphi(x) \rangle = \frac{\int \mathcal{D}\varphi \, e^{i/\hbar \, S(\varphi)} \varphi(x)}{I}$$

$$\approx$$

$$\varphi_{cl}(x) + \hbar\text{-corrections}$$

Stationary phase approximation: in limit $\hbar \to 0$ integral concentrates on locus φ_{cl} solving classical eqn

Problems with this formulation – eg defining theories with spin ....

Instead of defining I we'll formulate its properties & define an object satisfying these properties.

Separate two aspects of the problem:

1. $\hbar \to 0$ limit
2. dependence on space-time

Problem 2: recall that the action is <u>local</u> in origin: $S(\varphi) = \int_{\Sigma} \mathcal{L}(\varphi)$ where Lagrangian density is a <u>local</u> expression on spacetime Σ.

Fields on $\Sigma_1 \cup \Sigma_2$ = Fields on Σ_1 $\underset{\substack{\text{over common}\\ \text{boundary conditions}}}{\times}$ Fields on Σ_2.

$$\Rightarrow \int \mathcal{D}\varphi \; e^{\frac{i}{\hbar}\int_{\Sigma_1 \cup \Sigma_2} \mathcal{L}(\varphi)} =$$

$$= \int \mathcal{D}\varphi_1 \, \delta(\varphi_1 - \varphi_2) \int_{\varphi|_{\partial\Sigma_1} = \varphi_1} \mathcal{D}\varphi \, e^{\frac{i}{\hbar}\int_{\Sigma_1} \mathcal{L}(\varphi)} \int_{\varphi|_{\partial\Sigma_2} = \varphi_2} \mathcal{D}\varphi \, e^{\frac{i}{\hbar}\int_{\Sigma_2} \mathcal{L}(\varphi)}$$

So $\int \mathcal{D}\varphi\, e^{\frac{1}{\hbar}\int_\Sigma \mathcal{L}(\varphi)} \in W_1 \otimes \ldots \otimes W_k$

where W_i = space of functions on the space of φ_i:

$\partial\Sigma = \bigcup_{i=1}^{k} \Gamma_k$, $\varphi|_{\Gamma_k} = \varphi_k$

Sewing axiom: $I(\Sigma) \in W_1 \otimes \ldots \otimes W_k$

$I(\Sigma_1 \cup \Sigma_2) = \langle S / I(\Sigma_1) \otimes I(\Sigma_2) \rangle$

with sewing covector $S \in W_1^* \otimes W_2^*$.

Deformations of functional integral are internal:

can change Lagrangian $\mathcal{L}(\varphi) \rightsquigarrow \mathcal{L}(\varphi) + \delta\mathcal{L}(\varphi)$

$e^{\int_\Sigma \mathcal{L} + \delta\mathcal{L}} = e^{\int \mathcal{L}} \cdot \int_\Sigma \delta\mathcal{L}(\varphi)$ to first order in δ

so $\delta I = \int \mathcal{D}\varphi\, e^{\int_\Sigma \mathcal{L}} \int_\Sigma \delta\mathcal{L}$

deformation of a functional integral is a correlator!

eg $\int_{\text{metric}} I \simeq \int_{\Sigma} dx \int \mathcal{D}\varphi \, e^{\int_\Sigma \mathcal{L}} \int_{\text{metric}} \mathcal{L}(\varphi)(x)$

So need to consider also space of local observables V at marked points on Σ, in addition to space W of boundary condition

$\Rightarrow$ geometric data: Σ with k boundaries and n marked points

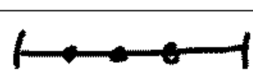
$$I(\Sigma_n) \in W^{\otimes k} \otimes V^{\otimes n}$$

Factorization axiom:

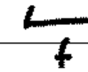
$$I(\Sigma_{n_1} \cup \Sigma_{n_2}) = \langle S | I(\Sigma_{n_1}) \otimes I(\Sigma_{n_2}) \rangle$$

Can we construct examples?

Simplest space-times: just time (0+1 dim)

$\Sigma =$  or 

$V = \mathrm{End}(W)$.

Definition of Hamiltonian:  $\longleftrightarrow e^{-tH}$

comes from derivative of semigroup of intervals

[Assume W has a scalar product]

$$I\left(\overset{t_1 \;\; t_2}{\vdash\!\bullet\!-\!\bullet\!-\!\cdots\!-\!\dashv}\right) = e^{-t_1 H} \varphi_1 e^{-t_2 H} \cdots\cdots \varphi_n e^{-t_n H}$$

with $\varphi_i \in \operatorname{End} W$: standard operator formulation of quantum mechanics.

...... representation of axioms of QFT in $d=1$.

Problems:

1. Construct QFTs in higher dimension
2. Find $\hbar$, and $\hbar \to 0 \rightsquigarrow$ classical physics (= classical geometry)

Original QFT has no parameter $\hbar$, which comes from trying to describe a classical limit — but can have many classical limits.

Analogy: a modular form has asymptotics (q-expansion) with meaningful coefficients .... but could have many cusps $\Rightarrow$ many expansions $\Longrightarrow$ many problems to which this may relate. Dualities: relate these different expansions of the same object.

$\underline{d=2}$ Construct QFT from chiral algebras associated to loop algebras

Σ Riemann surface with k boundaries

Pick k representations of the chiral algebra

I is then a solution to equations $\int_{\partial\Sigma}(f\cdot j)|I\rangle = 0$

f holomorphic function on Σ

j current, $f_j \in$ loop algebra

$\int_{\Gamma_i} f\cdot j \ : W_i \circlearrowleft$ $\qquad$ Γ_i component of $\partial\Sigma$

So I is an invariant of some algebra.

This space of sols is finite dimensional in some good cases

Current algebra $[j^a(\sigma), j^b(\sigma')] =$

$$= f^{ab}_c \, \delta(\sigma-\sigma') \, j^c(\sigma)$$

$$+ \, k \, \delta^{ab} \, \delta'(\sigma-\sigma')$$

(affine Kac-Moody algebra)

$\int_{\Gamma} j^a(\sigma) f(\sigma) \in KM$ algebra

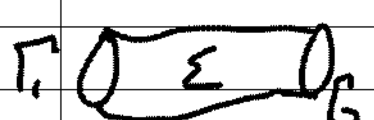
$|I\rangle \in W_1 \otimes \dots \otimes W_k$ $\qquad \eta^{\alpha\beta}$ bilinear form

Special property: $\eta^{\alpha\beta} |I_\alpha\rangle \otimes |\overline{I_\beta}\rangle$ has zero monodromy over space of complex structures (soln of KZ)

<u>Gaussian QFT</u> Suppose X has a linear structure

Consider maps $\varphi : \Sigma \longrightarrow X$ with action $S(\varphi)$ which is <u>quadratic</u> (wrt linear structure).

$\Longrightarrow$ can calculate functional integrals:



Write $\varphi = \varphi_{cl} + \varphi_{quantum}$ with φ_{cl} solution of equations of motion with given boundary conditions

& $\varphi_{quantum}$ has <u>zero</u> boundary conditions

$$\Rightarrow \int \mathcal{D}\varphi \, e^{\frac{1}{\hbar} S(\varphi)} =$$

$$= e^{\frac{1}{\hbar} S(\varphi_{cl})} \int \mathcal{D}\varphi_{quantum} \, e^{\frac{1}{\hbar} S(\varphi_{quantum})}$$

... we have a fixed boundary condition & pick a classical solution φ_{cl} with this boundary condition:

$$S(\varphi_{cl} + \varphi_q) = S(\varphi_{cl}) + \cancel{\frac{\delta S}{\delta \varphi} \varphi_q} + S(\varphi_q)$$

So dependence on boundary conditions (& almost everything) is captured by $e^{\frac{1}{\hbar} S(\varphi_{cl})}$

[eg $X = \mathbb{R}$: $S(\varphi) = \int_\Sigma d\varphi * d\varphi$]

Get equations on determinants

[eg $\int \mathcal{D}\varphi_q \, e^{-\int d\varphi_q * d\varphi_q} = (\det \Delta)^{-\frac{1}{2}}$]

Δ = Laplacian on space of φ vanishing at the boundary

T-duality: $X = S^1$ of radius R

$\varphi : \Sigma \longrightarrow \mathbb{R}$ with $\mathbb{Z}$ discontinuities

(think of as map to $\mathbb{R}/\mathbb{Z}$)

or as multivalued function, $d\varphi$ well defined

$$S = -R^2 \int_\Sigma d\varphi \, {*d\varphi}$$

Idea: represent action

$$\int \mathcal{D}\varphi \, e^{S(\varphi)} = \int \mathcal{D}p \, \mathcal{D}\varphi \, e^{i\int p\, d\varphi - \frac{1}{R^2} p \wedge * p}$$

with p a 1-form on Σ ...

this is a Gaussian integral $\int \mathcal{D}p \, e^{i\int p\, d\varphi - \frac{1}{R^2} p * p}$

.... complete the square

$$\int dp \; e^{ipx - \frac{1}{2R^2} p^2} = \int dp \, e^{-\frac{1}{2}\left(\frac{p}{R} + ixR\right)^2} e^{-\frac{x^2 R^2}{2}}$$

— recover original action

If φ is single-valued $\Rightarrow$ $ip\, d\varphi \rightsquigarrow -i\varphi\, dp$

so integral $\int d\varphi \, e^{-i\varphi\, dp} = \delta(dp)$

So $d\rho = 0$, can write $\rho = d\varphi_D$ (ignoring cohomology)

So our integral becomes

$$\int \mathcal{D}\varphi_D \, e^{-\frac{1}{2R^2}\int d\varphi_D \wedge * d\varphi_D}$$

We've replaced R^2 by $\frac{1}{R^2}$!

Introduce auxiliary ρ, change order of integration, find ρ is exact $\Rightarrow$ write it as d(new variable φ_D).

Careful analysis shows φ valued in $\mathbb{R}/R\mathbb{Z}$

$\Longrightarrow$ φ_D valued in $\mathbb{R}/\frac{1}{R}\mathbb{Z}$.

Now look at correlators instead:

correlator of ρ

$$\int \mathcal{D}\varphi \, \mathcal{D}\rho \; \rho \; e^{S(\varphi,\rho)}$$

: eqn of motion from completing square is

$\rho = R^2 * d\varphi$

But $\rho = d\varphi_D \Rightarrow \boxed{\langle d\varphi_D \rangle = R^2 \langle * d\varphi \rangle}$

$\Rightarrow$ relation between vortices & tachyons