Ivan Mirkovic - Langlands in positive characteristic

Note Title 1/18/2007

(Nonexistent) science of critical quantization:
complications come from (naive) failures of quantization
leaving large commutative pieces

Today: $\mathcal{D}_X$ in char $p>0$

$$\text{reps of Lie algebras} \longleftarrow \mathcal{D}_X \longrightarrow GL \text{ in positive char}$$

& quantum groups at roots of unity,
critical level reps, super Lie algebras
.... some such formalism should exist!
— family of theories degenerating at
particular values to being "almost commutative"
... "much worse" at these values,
but all complications come from the center.

$\mathcal{D}_X$: family over characteristics: generically
in char 0 $\mathcal{D}_X$ is simple,
but if $k=\bar{k}$ characteristic $p>0$
$\Rightarrow Z(\mathcal{D}_X)$ large

$$\mathcal{D}_X := \langle \mathcal{O}_X, T_X \rangle / \text{usual relations} \cong \bigoplus \mathcal{O}_X \cdot \partial^I$$

– "crystalline" differential operators:
the honest differential operators have divided powers "$\frac{\partial^n}{n!}$".

Case $X = \mathbb{A}^1$: $\mathcal{D}_X = \bigoplus k(x^i \partial^j)$

$[\partial, x] = 1$, but $[\partial, x^p] = p x^{p-1} = 0$,
$[\partial^p, x] = 0$

$\Rightarrow$ center $Z(\mathcal{D}_X) = k[x^p, \partial^p] = \mathcal{O}(T^*X)^p$
p^{th} powers of functions on the cotangent
(raising to p^{th} power is an algebra map, $(a+b)^p = a^p + b^p$)

Y variety
$\mathcal{O}(Y) \supset \mathcal{O}(Y)^p = \mathcal{O}(Y^{(1)})$
p^{th} power functions are functions on the Frobenius twist
$\Rightarrow$ map $Y \xrightarrow{Fr} Y^{(1)}$

Lemma a. $Z(\mathcal{D}_X) = \mathcal{O}(T^*X^{(1)})$

b. $\mathcal{D}_X$ is an Azumaya algebra over its center, i.e if $p \in (T^*X)^{(1)}$ $\Rightarrow$ the specialization

$(\mathcal{D}_X)_p \simeq M_n(k)$ matrix algebra

$n = p^{\dim X}$

[but matrix algebras have zero divisors & $\mathcal{D}_X$ doesn't, so must be twisted!]

— Sheaf A of algebras over X is Azumaya if it is locally End of a vector bundle V

(étale or flat or smooth, not Zariski)

we then say V is a (local) splitting of A (over open U) $\Longrightarrow$ $A|_U\text{-mod} \simeq \mathrm{Coh}\, U$

$$\mathcal{F} \otimes V \longleftarrow \mathcal{F}$$

so where A splits the noncommutative world looks commutative!

e.g $(0,0) \in T^*\mathbb{A}^1$

$$(\mathcal{D})_{(0,0)} = \mathcal{D}_X / x^p = 0 = \partial^p = \bigoplus_{0 < i,j < p} k\, x^i \partial^j$$

This algebra $(\mathcal{D}_X)_{(0,0)}$ acts on $(\mathcal{O}_X)_{0 \to (0,0)} = k[x]/x^p$

<u>Lemma, cont.</u> :

C. $\mathcal{D}_X$ splits on some Lagrangians (eg conormals to smooth subvarieties)

... e.g on 0-section $T^*_X X$: $\mathcal{D}_X|_{T^*_X(X)} \simeq \mathcal{E}nd_{\mathcal{O}_X^{(1)}} \mathcal{O}_X$

(rank $p^{\dim X}$ vector bundle on $X^{(1)}$).

.... this is a gerbe used by Deligne–Illusie in their work on Hodge–de Rham.

"Abelianization" : X space, gerbe $\mathcal{G}$ on X : sheaf of categories .....

eg BG = category of G-torsors,

a G-gerbe on X is a sheaf of categories locally $\simeq$ category of G-torsors. ($\underset{\mathrm{Tors}_G}{\text{torsor over}}$)

eg $\mathbb{G}_m$ case $\mathcal{G}$ locally $X \times B\mathbb{G}_m$

$\mathcal{C}oh(\mathcal{G}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}oh(\mathcal{G})_n$: locally graded coherent sheaves.

Azumaya algebra $A \rightsquigarrow$ gerbe $\mathcal{A}$

$U \subset X$, $\mathcal{A}(U) =$ splittings of $A|_U$:

for any two splittings V', V'' have $V' \simeq L \otimes V''$

L a line bundle.

Modules over an Azumaya algebra

$\mathcal{M}(A) = \mathcal{C}oh(\mathcal{A})_1$, degree 1 sheaves.

$\mathcal{D}_X \rightsquigarrow$ gerbe $\mathbb{D}_X$, $\mathcal{D}_X$-mod are $\mathcal{C}oh(\mathbb{D}_X)_1$.

Geometric Langlands: (unramified, GL_n, $p>0$)

Classical picture:

$$\begin{tikzcd} T^*Bun_G \arrow[dr] & & T^*Bun_{G^\vee} \arrow[dl] \\ & H_G \simeq H_{G^\vee} & \end{tikzcd}$$

dual abelian varieties over regular locus:

$H_G \supset H_r$ regular locus

$T^*Bun_G|_{H_r} = Pic(C_H)$, $C_H \to H$ spectral curve.

I. Strategy: Fourier-Mukai

II. Deform both sides:

$T^*\mathrm{Bun}_G \rightsquigarrow \mathcal{D}_{\mathrm{Bun}_G}$ noncommutative deformation

$T^*\mathrm{Bun}_{G^\vee} \rightsquigarrow \mathcal{L}oc_{G^\vee}$ torsor for $T^*\mathrm{Bun}_{G^\vee}$
$$\downarrow$$
$$\mathrm{Bun}_{G^\vee}$$

Kapustin-Witten: have hyperkähler
structure on $T^*\mathrm{Bun}_G$ including the deformation.
(h-k structure should exist almost all p ??)

Bezrukavnikov-Braverman: $p>0$ $\underline{G=GL_n}$

$D^b[\mathrm{Bun}_G, \mathcal{D}] = D^b[\mathbb{D}]_1$ coherent sheaves on gerbe

The gerbe $\mathbb{D}$ is actually grouplike:

compatible with the group structure on the
Hitchin fibration : $m^*A \simeq A \boxtimes A$

$\mathbb{D} / T^*\mathrm{Bun}_G^{(1)}$ grouplike: abelian group struct,
which is an extension
$$0 \to B\mathbb{G}_m \to \mathbb{D} \to T^*\mathrm{Bun}_G^{(1)} \to 0$$

Cartier duality: $\mathcal{C}(A) = R\mathrm{Hom}(A, \mathbb{G}_m)[1]$
A abelian group

e.g. $\mathcal{C}(\mathbb{Z}) = B\mathbb{G}_m$. $\mathcal{C}^2 = id$

$\mathcal{C}(A) = A^\vee$ for abelian varieties.

Dualize $\mathbb{D}$:
$$\begin{tikzcd}
0 \arrow[r] & T^*\mathrm{Bun}^{(1)}_{G^\vee} \arrow[r] & \mathcal{C}(\mathbb{D}) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & \mathcal{C}(\mathbb{D})_1 \arrow[u, hook] \arrow[r] & 1 \arrow[u, hook] &
\end{tikzcd}$$

torsor for T^*.

Fourier–Mukai/Pontrjagin duality:
$$D^b(\mathbb{D}, \mathcal{O}) \simeq D^b(\mathcal{C}\mathbb{D}, \mathcal{O})$$
$$D^b(\mathbb{D}, \mathcal{O})_1 \simeq D^b([\mathcal{C}\mathbb{D}]_1, \mathcal{O})$$

Now just need to compare torsors:
$$[\mathcal{C}\mathbb{D}]_1 \simeq \mathcal{LS}_{G^\vee}:$$

$$\begin{tikzcd}
\mathcal{LS}_{G^\vee} \arrow[dr] & & T^*\mathrm{Bun}^{(1)}_{G^\vee} \arrow[dl] \\
 & H^{(1)}_{G^\vee} &
\end{tikzcd}$$

: local systems are torsor over $T^*\mathrm{Bun}^{(1)}_{G^\vee}$ over the (twisted) Hitchin base

D_X-module $\mathcal{F}$: p-curvature $\psi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X^{(1)}}$
($\Leftrightarrow$ comes from sheafification of $\mathcal{F}$
over $T^*X^{(1)}$.)

$\psi \in \Gamma(X, \mathcal{E}nd\, \mathcal{F} \otimes \Omega^1_{X^{(1)}})$
$\Rightarrow$ get Hitchin-p-curvature map
$$\begin{array}{c} \mathcal{L}S_{G^\vee} \\ \downarrow \\ H^{(1)}_{G^\vee} \end{array}.$$

Representations of Lie algebras in Char $p > 0$.
$\rightsquigarrow$ additional structure on $D^b Coh(T^*\mathcal{B})$.
abelian subcategory $\mathcal{E}$ of exotic sheaves.
($D^b Coh = D^b \mathcal{E}$).

G semisimple group $/k$, $p > h$ Coxeter number.

$\mathrm{Rep}\, \mathfrak{g} = \mathcal{M}(U\mathfrak{g})$ modules
$Z(U\mathfrak{g}) \supset \mathcal{O}(\mathfrak{h}^*/W) = \underset{\text{Harish-Chandra}}{Z(U\mathfrak{g})_{HC}}$

but have bigger center:

$$Z(U\mathfrak{g}) = Z(\mathcal{D}(G)^{1\times G}] \supseteq \mathcal{O}(T^*G^{(1)})^{1\times G}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}(\mathfrak{g}^{*(1)})$ p-center

in fact

$$Z(U\mathfrak{g}) = Z_{HC} \otimes \mathcal{O}(\mathfrak{g}^{*(1)}).$$

So for $\lambda \in \mathfrak{h}^*$ $\chi \in \mathfrak{g}^{*(1)}$ can specialize:

$(U\mathfrak{g})^\lambda_\chi = U^\lambda_\chi$ finite dimensional.

$\mathcal{B}$ = flag variety of G. $\mathfrak{g} \to \mathcal{T}_\mathcal{B}$, $U\mathfrak{g} \to \mathcal{D}_\mathcal{B}$

Beilinson-Bernstein localization:

$$\Delta: M \longmapsto \mathcal{D}_\mathcal{B} \overset{\mathbb{L}}{\underset{U}{\otimes}} M \quad ,$$

$$D^b(U\mathfrak{g}\text{-mod}) \longrightarrow D^b(\mathcal{D}_\mathcal{B}\text{-mod})$$

<u>**Theorem**</u> (Bezrukavnikov-Mirkovic-Rumynin)

$$\begin{array}{ccc} \Delta: D^b[U^{\lambda=0}] & \xrightarrow{\sim} & D^b(\mathcal{D}_\mathcal{B}) \\ \uparrow & & \uparrow \\ D^b_\chi[U^0] & \xrightarrow{\sim} & D^b_{\mathcal{B}^{(1)}_\chi}(\mathcal{D}_\mathcal{B}) \end{array}$$

$\chi \in N \subset \mathfrak{g}^*$ nilpotent

Note $D^b(\mathcal{D}_{\mathcal{B}})$ is Calabi-Yau.
[Bondal-Kapranov ?] : CY triangulated categories don't have retracts!

we know
$$\begin{tikzcd} D^b\mathcal{U}^0 \arrow[r] \arrow[d, "id"] & D^b(\mathcal{D}_{\mathcal{B}}) \arrow[dl] \\ D^b(\mathcal{U}) & \end{tikzcd}$$

(in char 0 it's not exactly Calabi-Yau ...)

$\mathcal{B}_\chi^{(1)}$: Springer fiber over χ.

$D^b_{\mathcal{B}_\chi^{(1)}}(\mathcal{D}_{\mathcal{B}})$: supported on Springer fiber.

<u>Claim</u> $\mathbb{D}$ <u>splits</u> on formal neighborhood of Springer fibers $\mathcal{B}_\chi^{(1)}$

$$\Rightarrow D^b_\chi[\mathcal{U}^0] \simeq D^b[\mathcal{C}oh_{\mathcal{B}_\chi^{(1)}}(T^*\mathcal{B}^{(1)})]$$

set theoretic support.

Representation theory & Algebraic Geometry

we find $D^b[\mathrm{Coh}_{\mathcal{B}_\chi^{(1)}} T^*\mathcal{B}^{(1)}]$ carries a t-structure coming from U_χ^0-modules

... <u>exotic sheaves</u>.

In fact this t-structure is defined on all of $D^b[\mathrm{Coh}\ T^*\mathcal{B}]$

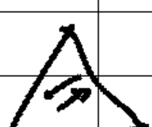
$$\Rightarrow \quad K(\mathcal{B}_\chi) = \mathbb{Z}[\mathrm{Irr}(U_\chi^0)]$$

Canonical basis in K-group of $\mathcal{B}_\chi$ (characterized modulo existence by Lusztig — so now we get existence!)

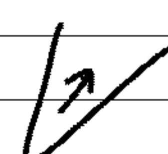
Action of affine braid group:

in representation theory given by intertwining functors: translation functors + Hecke:

move to walls & back in Weyl alcove



In char 0 get Weyl <u>chamber</u>, so get finite braid group.



Algebro-geometric picture: explored in Bezrukavnikov's talk.

<u>Exotic sheaves</u>: consider $T^*\mathcal{B} \xrightarrow{\pi} \mathcal{N}$ Springer map.

A Exotic $\Longleftrightarrow$ $(*)$ $\begin{cases} \bullet \ \pi_* A \text{ is exact (a sheaf)} \\ \bullet \ I_w A \in D^b(\tilde{\mathcal{N}})^{\leq 0} \end{cases}$

I_w intertwiner for $w \in W_{aff} \hookrightarrow B_{aff}$

Equivalently A is in degree ≤ 0 for exotic structure
$\Longleftrightarrow$ $\pi_* (I_w A) \leq 0$ on $\mathcal{N}$ $\forall w \in B_{aff}$
(<u>definition</u> of exotic sheaves).

$(*)$ suggests relation to critical level:
this is how one characterizes critical level
.... applying intertwining functor goes only down
- at positive level everything seen in pushforward from flags to $\hat{\mathfrak{g}}$-modules, but

above are precisely critical level characters.

$\mathfrak{sl}_2$ case : these are Bridgeland's perverse coherent sheaves for a flop.
(shift those who die in direct image to $\mathcal{U}$)

Relation to geometric Langlands

(1) t-structures on $T^*\mathcal{B}$ & its extended version $\widetilde{T^*\mathcal{B}} \longrightarrow \mathfrak{g}$. Can restrict to inverse image of any $Y \subset \mathfrak{g}$
$Coh(\widetilde{T^*\mathcal{B}}, \widetilde{Y})$.

If we double $\mathfrak{g}$ get $Coh(St)$
Steinberg variety

(2) $D^b[Coh_G(T^*\mathcal{B})] \underset{\substack{\text{local tamely} \\ \text{ramified GL}}}{\sim} D^b_I({}^LG((t))/{}^LG(0))$
(Bezrukavnikov

"perverse exotic" t-structure : shift along orbits in $\mathcal{N}$ $\longleftarrow$ perverse t-structure

In $\mathcal{P}_I({}^LG(\mathcal{K}) / {}^LG(\mathcal{O}))$ have canonical basis of objects $\Rightarrow$ can use them to define full exotic structures

Nonequivariant Coh on $T^*\mathcal{B}$:
Should correspond to critical level rep theory, while G-equivariants can be described perversely as above.

Ideas
3 Exotic structure on $D^b(Coh_{G(\mathcal{O})} T^* Gr_G)$

... should define by using double affine intertwining operators...

$D^b Coh(St_{Gr}) \supseteq$ Exotic perverse sheaves

Dyonic Wilson-'t Hooft operators $\circlearrowleft$ $D^b Coh T^* Bun_G$ $\Longleftrightarrow$ canonical basis of irreds labelled by double affine Weyl group

Canonical basis in Cherednik

Braverman–Finkelberg: look at some Uhlenbeck spaces, look like open cell of double loop Grassmannian, with opposite stratification, stalks of IC sheaves look like representations of loop groups.

(have a double loop Grassmannian for each level...)