Takuro Mochizuki - Twistor structure & harmonic bundles

Note Title 1/12/2007

X complex projective variety

$\uparrow \pi$

$T^*X \supset L$ Lagrangian, proper

$(\mathcal{L}, \nabla)$ flat unitary line bundle on L

Problem: Construct a regular holonomic $\mathcal{D}$-module on X with some nice properties

$\Sigma \subset X$ locus s.t. $\pi' : L \setminus \pi^{-1}\Sigma \longrightarrow X \setminus \Sigma$ is a covering

$\Rightarrow (V, \nabla) := \pi'_* (\mathcal{L}, \nabla)\big|_{L \setminus \pi^{-1}\Sigma}$ is a flat unitary bundle on $X \setminus \Sigma$.

$$\begin{array}{ccc} \varphi : \tilde{X} & \longrightarrow & X \\ \cup & & \cup \\ \tilde{D} = \varphi^{-1}(\Sigma) & \longrightarrow & \Sigma \end{array}$$

normal crossing

$\Longrightarrow$ Deligne extension $(\tilde{V}, \nabla)$ over $\tilde{X}$ with regular singularities

(f holomorphic section of $\tilde{V} \Rightarrow$
$\nabla f \in \tilde{V} \otimes \Omega^1(\log \tilde{D})$)

& $\operatorname{Res} \tilde{\nabla} \curvearrowright \tilde{V}|_{\tilde{D}}$ has eigenvalues

$0 \leq \mathrm{Re}\left(\begin{smallmatrix}\text{eigenvalues}\\ \text{of } \mathrm{Res}(D)\end{smallmatrix}\right) < 1$

$\exists$ natural parabolic structure F for $\tilde{V}$

s.t. parabolic $c_1(\tilde{V}, F) = 0$

" $ch_2(\tilde{V}, F) = 0$

$\theta_{\tilde{V}}$ Higgs field of $\tilde{V}$ given by Lagrangian L on $X \setminus \Sigma \overset{\text{exact}}{\Longrightarrow}$

$\exists\ \tilde{\theta}$ Higgs field of $(\tilde{V}, F)$ on $\tilde{X}$

& $(\tilde{V}, F, \tilde{\theta})$ satisfy stability condition

$\Longrightarrow$ $\exists\ h$ pluriharmonic metric for $(\tilde{V}, F, \tilde{\theta})$

Kobayashi-Hitchin for tame harmonic bundles

~~too make~~

$\Longrightarrow$ regular holonomic $\mathcal{D}$-module on X

$\mathcal{L}$ ample line bundle on $\tilde{X}$ $\Rightarrow$

$$\text{par-deg}_{\mathcal{L}}(\tilde{V}, F) := \int_{\tilde{X}} \text{par-}c_1(\tilde{V}, F)\, c_1(\mathcal{L})^{\dim \tilde{X}-1}$$

st for subsheaf $(F, \theta_F) \subset (\tilde{V}, \tilde{\theta})$

$$\frac{\text{par deg}_{\mathcal{L}}(F)}{\text{rk } F} < \frac{\text{par deg}_{\mathcal{L}}(\tilde{V})}{\text{rk } \tilde{V}}$$

3 parts

1. Mixed twistor structure MTS
   vs. Mixed Hodge structure MHS

2. harmonic bundles & limiting MTS.

3. Application to $\mathcal{D}$-module

Most ideas are due to Simpson

<u>Def</u> A pure twistor structure of weight n

$\Longleftrightarrow$ $V/\mathbb{P}^1(\mathbb{C})$ s.t $V \simeq \mathcal{O}_{\mathbb{P}^1}(n)^{\oplus r}$

Mixed twistor structure $\Longleftrightarrow$ $(V, W)_{\mathbb{P}^1}$

V vector bundle, W filtration $\subset W_n \subset W_{n+1} \subset \cdots$

s.t $Gr_n^W(V)$ is a pure twistor structure of wt n.

Fix $0, 1, \infty \in \mathbb{P}^1$

MTS : structure on fiber $V|_1$

<u>Recall</u> : pure Hodge structure of weight n

$\Longleftrightarrow$ (H, F, G) $\supset F^i \supset F^{i+1} \supset \cdots$ decreasing

$G^j \supset G^{j+1} \supset \cdots$

s.t. $H = \bigoplus_{i+j=n} F^i \cap G^j$

Mixed Hodge structure: (H, W, F, G) $W_n \subset W_{n+1} \subset \dots$ increasing

s.t. $(Gr^W H, F, G)$ PHS of wt n.

$\mathbb{C}^* \curvearrowright \mathbb{P}^1(\mathbb{C})$ $\quad t \cdot [z_0 : z_1] = [tz_0, z_1]$

Can look at $\mathbb{C}^*$-equivariant MTS

<u>Prop</u> (Simpson): $\begin{pmatrix} \text{equivariant} \\ \text{MTS} \end{pmatrix} \longleftrightarrow (\text{MHS})$

<u>Recall:</u> PHS $\Rightarrow$ equivariant PTS:

$(H, F, G) \Rightarrow H \otimes \mathcal{O}_{\mathbb{C}^*}$ vector bundle

on $\mathbb{C}^*_\lambda = \operatorname{Spec} \mathbb{C}[\lambda, \lambda^{-1}]$.

Filtration $F \Rightarrow$ via Rees construction get vector bundle $\xi(H,F)$ on $\mathbb{C}_\lambda = \operatorname{Spec} \mathbb{C}[\lambda]$

$\xi(H,F) = \sum \mathbb{C}[\lambda]\, \lambda^{-i} F^i$

$G \Rightarrow \xi(H, G)$ on $\mathbb{C}_{\lambda^{-1}} = \operatorname{Spec} \mathbb{C}[\lambda^{-1}]$

Glue together to $\xi(H,F,G)$ equivariant vector bundle $/ \mathbb{C}$

<u>Example</u> $H = \mathbb{C}$ $\quad F^p = H \supset F^{p+1} = 0$

$\qquad\qquad\qquad\qquad G^q = H \supset G^{q+1} = 0$

$\xi(H,F) = \mathbb{C}[\lambda] F^p \lambda^{-p} = \mathcal{O}_{\mathbb{C}_\lambda}(p \cdot 0)$

$\xi(H,G) = \mathcal{O}_{\mathbb{C}_{\lambda^{-1}}}(q \cdot \infty)$

$\xi(H,F,G) = \mathcal{O}_{\mathbb{P}^1}(p \cdot 0 + q \cdot \infty) \simeq \mathcal{O}_{\mathbb{P}^1}(p+q)$

$H = \bigoplus_{i+j=n} F^i \cap G^j \Rightarrow \xi(H,F,G) = \mathcal{O}(n)^{\oplus r}$

PTS of weight n.

So twistor structures <u>underly</u> Hodge structures (forget $\mathbb{C}^*$-equivariance)

<u>Principle</u> (Simpson's Meta Theorem)

Some claim holds for Hodge $\Rightarrow$ it holds for Twistor

<u>Example</u> (degeneration of spectral sequence)

$(V_\cdot, W)$ $\qquad \cdots \to (V_i, W) \to (V_{i+1}, W) \to \cdots$

complex of MTS $\qquad$ ($f: V_i \to V_{i+1}$ preserve filtration)

$\Rightarrow \quad E^1_{pq} = H_{p+q}(Gr^W_p(V_\cdot)) \Longrightarrow E^\infty_{pq} =$

$Gr^W_p H_{p+q}(V_\cdot)$

use $Gr^W_p(V_\cdot)$: $\quad \cdots \to Gr^W_p(V_i) \to Gr^W_p(V_{i+1}) \to \cdots$

[note : f morphism of PTS $\Rightarrow$ ker, coker,
Im are PTS of same weight]

E^1_{pq} PTS of weight p $\Longrightarrow$

$d : E^1_{pq} \longrightarrow E^1_{p+1, q+2}$ automatically

trivial : $\quad Hom(\mathcal{O}_{\mathbb{P}^1}(i), \mathcal{O}_{\mathbb{P}^1}(j)) = 0 \quad i > j$

$\Rightarrow E^1_{pq} = E^\infty_{pq}$.

<u>In particular $(H(V_\cdot), W)$ is an MTS.</u>

<u>Tate twist</u> : $\quad \mathbb{T}(n) = \mathcal{O}_{\mathbb{P}^1}(-2n)$

$(V_\cdot, W) \mapsto (V, W) \otimes \mathbb{T}(n)$

<u>Complex conjugate</u> : Hodge : $\quad (H, F, G) \mapsto (\overline{H}, G, F)$

PTS case: V PTS of weight n

$\sigma : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad \sigma([z_0, z_1]) = [-\bar{z}_1, \bar{z}_0]$

$\sigma^* V$ PTS of weight

$f \cdot \sigma^* (v) = \sigma^* (\overline{\sigma^*(f)} \cdot v)$

Fix $\quad \iota : \mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\sim} \sigma^* \mathcal{O}_{\mathbb{P}^1}(1)$

s.t. $\quad \sigma^*(\iota) \circ \iota = -1$

$\Rightarrow \quad \mathcal{O}_{\mathbb{P}^1}(n) \simeq \sigma^* \mathcal{O}_{\mathbb{P}^1}(n)$

$\mathbb{T}_{\mathbb{P}^1}(n) \simeq \sigma^* \mathbb{T}(n)$

<u>Def</u> Polarization of PTS V of weight 0

$\Leftrightarrow \quad S : V \otimes \sigma^* V \longrightarrow \mathbb{T}(0)$ "symmetric",

$H^0(S)$: induced hermitian pairing on $H^0(V)$ is positive definite

V weight w $\quad S$ polarization of V

$\Longleftrightarrow \quad S : V \otimes \sigma^*(V) \longrightarrow \mathbb{T}(-w) \quad (-1)^w$-symmetric

$S : V \otimes \mathcal{O}_{\mathbb{P}^1}(-w) \otimes \sigma^*(V \otimes \mathcal{O}(-w)) \longrightarrow \mathbb{T}(0)$

polarization of $V \otimes \mathcal{O}_{\mathbb{P}^1}(-w)$

$\rightsquigarrow$ polarized MTS

X complex manifold

$(E, \bar\partial_E, \theta)$ Higgs bundle $/X$

h hermitian metric on E

$\Rightarrow \quad \partial_E, \quad \bar\partial_h(u,v) = h(\bar\partial_E u, v) + h(u, \partial_E v)$

Set $\theta^\dagger$ adjoint, ie $h(\theta u, v) = h(u, \theta^\dagger v)$

$\rightsquigarrow \quad \mathbb{D}' := \bar\partial_E + \theta^\dagger + \partial_E + \theta$

Def h is pluriharmonic $\Longleftrightarrow \mathbb{D}' \circ \mathbb{D}' = 0$

$\rightarrow E, \bar\partial_E, h, \theta$ harmonic bundle.

Note $\forall \lambda \in \mathbb{C} \quad \mathbb{D}^\lambda := \bar\partial_E + \lambda\theta^\dagger + \lambda\partial_E + \theta$

λ-connection, $\quad \mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0 \quad \forall \lambda \in \mathbb{C}$.

$[\mathbb{D}^\lambda(f \cdot s) = (\bar\partial_X + \lambda\partial_X) f \cdot s + f\,\mathbb{D}^\lambda s \quad]$

Prop (E, ∇) flat bundle / quasiprojective variety

E, ∇ semisimple $\Longleftrightarrow \exists$ pluriharmonic metric on (E, ∇) satisfying condition at ∞

[projective : Corlette, quasiprojective: mainly Jost - Zuo]

Example polarized variations of PHS

Prop (Simpson) $\begin{pmatrix} \text{harmonic} \\ \text{bundle} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{variation of} \\ \text{polarized} \\ \text{PTS of wt } w \end{pmatrix}$

$\mathcal{E}^\lambda = (E, \bar\partial_\lambda + \lambda\theta^+)$ holomorphic bundle on X
with $\mathbb{D}^\lambda$ λ-connection

$(E, \bar\partial_E, h, \theta)$ harmonic bundle $/X$
$\Longrightarrow (E, \partial_E, \theta^+)$ on X^+ complex conjugate

$\mu \in \mathbb{C} \Longrightarrow (E, \partial_E + \mu\theta)$
with $\mathbb{D}^{+\mu} = \partial_E + \mu\theta + \mu\bar\partial_E + \theta^+$

So have λ-family on $X \times \mathbb{C}_\lambda$
& μ-family on $X^+ \times \mathbb{C}_\mu$

For $\lambda \neq 0$ can identify λ- & μ-connections
for $\lambda = \mu^{-1}$:

$$\mathbb{D}^{\lambda f} = \bar\partial_E + \lambda\theta^\dagger + \partial_E + \lambda^{-1}\theta$$
$$= \bar\partial_E + \mu^{-1}\theta^\dagger + \partial_E + \mu\theta = \mathbb{D}^{\dagger\mu f}$$

$\Longrightarrow$ glue together to bundle on $X \times \mathbb{P}^1$.

Example! $X = \{ z : |z| < 1 \}$ $D = \{0\}$

$(E, \bar\partial_E, h, \theta)$ tame harmonic on $X \setminus D$

$\lambda \in \mathbb{C} \Longrightarrow \mathcal{E}^\lambda, \mathbb{D}^\lambda \,/\, X \setminus D$

$a \in \mathbb{R}$ $\quad {}_a\mathcal{E}^\lambda$: $\mathcal{O}_X$-module with property $U \subset X$ open:

$${}_a\mathcal{E}^\lambda(U) = \{ f \in \mathcal{E}^\lambda(U \setminus D) : |f|_h = O(|z|^{-a-\varepsilon}) \ \forall \varepsilon > 0 \}$$

<u>Prop</u> ${}_a\mathcal{E}^\lambda$ is coherent (in fact locally free) & $\mathbb{D}^\lambda$ has regular singularities

$$Gr_a^F(\mathcal{E}^\lambda) = {}_a\mathcal{E}^\lambda \Big/ \sum_{b<a} {}_b\mathcal{E}^\lambda$$

$$\mathrm{Res}(\mathbb{D}^\lambda) \circlearrowright Gr^F\, {}_a\mathcal{E}^\lambda = \bigoplus_{\alpha\in\mathbb{C}} Gr^{FE}_{a,\alpha}(\mathcal{E}^\lambda)$$

Kashiwara-Malgrange-Simpson

$$KMS(\mathcal{E}^\lambda) = \{(a,\alpha) : Gr^{FE}_{(a,\alpha)}(\mathcal{E}^\lambda) \neq 0\}$$

$$k(\lambda) : \mathbb{R}\times\mathbb{C} \xrightarrow{\sim} \mathbb{R}\times\mathbb{C}$$

$$k(\lambda, a, \alpha) = (a + 2\,\mathrm{Re}(\lambda\bar{\alpha}),\ \alpha - a\lambda - \bar{\alpha}\lambda^2)$$

<u>Prop</u> $k(\lambda) : KMS(\mathcal{E}^0) \xrightarrow{\sim} KMS(\mathcal{E}^\lambda)$

$$\mathcal{G}_u^\lambda(E) := Gr^{FE}_{k(\lambda,u)}(\mathcal{E}^\lambda)$$

$$\mathcal{G}_u(E) = \bigcup_{\lambda\in\mathbb{C}} \mathcal{G}_u^\lambda(E) \text{ hol. bundle on } \mathbb{C}_\lambda$$

$\lambda \neq 0$ $H(\mathcal{E}^\lambda) :=$ multivalued flat sections

Def: $s \in F_a H(\mathcal{E}^\lambda)$ if $\left| s|_{\gamma(t)} \right|_h = O(t^{-a-\varepsilon})\ \forall \varepsilon > 0$
$\gamma(t) \to t_0$

$$\circlearrowright \; Gr^F_a \, H(\mathcal{E}^\lambda) = \bigoplus Gr^{FE}_{aw} \; H(\mathcal{E}^\lambda)$$

monodromy

$$KMS^+(\mathcal{E}^\lambda) := \{(a,w) : \dim^{FE}_{(a,w)} Gr\, H(\mathcal{E}^\lambda) \neq 0\}$$

$$k^+(\lambda) : (\mathbb{R}/\mathbb{Z}) \times \mathbb{C} \longrightarrow \mathbb{R} \times \mathbb{C}^*$$

$$k^+(\lambda, a, \alpha) = (Re(\lambda\bar{\alpha} + \lambda^{-1}\alpha) \exp(2\pi i \cdot (\lambda^{-1}\alpha - a - \lambda\bar{\alpha})))$$

<u>Prop</u> $k^+(\lambda) : KMS(\mathcal{E}^0)/\mathbb{Z} \simeq KMS^+(\mathcal{E}^\lambda)$

$$\mathcal{G}^\lambda_u H(E) = \bigcup_{\lambda \in \mathbb{C}^*_\lambda} \mathcal{G}^\lambda_u H(E)$$

$$\Phi : \mathcal{G}_u \mathcal{H}(E) \xrightarrow{\sim} \mathcal{G}_u(E)|_{\mathbb{C}^*_\lambda}$$

$\rightsquigarrow$ can glue $\mathcal{G}^+_{u^+}(E)$ to $\mathcal{G}_u(E)$

to obtain a holomorphic bundle

$$S_u(E)/\mathbb{P}^1$$

N nilpotent part of residue $\circlearrowright \mathcal{G}_u(E)$

$\rightsquigarrow N^{\Delta}: S_u(E) \longrightarrow S_u(E) \otimes \mathbb{T}(-1)$

$\Longrightarrow$ weight filtration W on $S_u(E)$ making $S_u(E), W$ an MTS: the <u>limit MTS</u>. (also have polarization!)

2-variable case:

$(S_u(E), N_1, N_2, S)$ Pol. MTS in 2 variables.

polarization

$(E, \bar{\partial}_E, h, \theta)$ on $(\Delta^*)^2$, $0 \in (\Delta^*)^2$

Filtration W is induced by nilpotent $N_1 + N_2$ [Denote $V = S_u(E)$]

$W(N_1)$ filtration induced by N_1

$\Longrightarrow$ 2 filtrations on $Gr^{W(N_1)}(V)$:

$W^{(1)}$ induced by W, $N_2^{(1)} \circlearrowright Gr^{W(N_1)}(V) \Rightarrow W(N_2^{(1)})$

$$W^{(1)}_{a+k} \cap Gr_a^{W(N_1)} = W(N_2^{(1)})_k \cap Gr_a^{W(N_1)}$$

compatibility of filtrations.

This is important for showing the norm estimates on holomorphic sections

(the harmonic metric is determined by Parabolic filtration of monodromy, up to boundedness)

E, ∇ flat on $X \setminus D$

want to construct pluriharmonic h.

Above story on asymptotic behavior is used to establish properties at ∞ of this correspondence. ...

$\mathcal{D}$-modules

$f: X \longrightarrow Y$ projective morphism of algebraic varieties $/\mathbb{C}$, L rel. ample bundle

M algebraic regular holonomic $\mathcal{D}$-mod$/X$, semisimple

$$(*) \begin{cases} \Rightarrow \mathcal{H}^i f_* M \text{ are semisimple} \\ c_1(\mathcal{L})^i : \mathcal{H}^{-i} f_* M \xrightarrow{\sim} \mathcal{H}^i f_* M \\ \text{(hard Lefschetz)} \end{cases}$$

Sabbah: defined pure twistor $\mathcal{D}$-modules
(twistor version of pure Hodge modules)

M comes from pure twistor $\mathcal{D}$-module, polarized

$\Longrightarrow$ $(*)$ ok:

Mochizuki: pure twistor $\mathcal{D}$-module $\longleftrightarrow$ semisimple reg. holonomic $\mathcal{D}$-modules

($(*)$ also proved by Drinfeld, Lafforgue, Gaitsgory, Boeckle - Kh.. by arithmetic methods.)

Higgs side: eigenvalues pure imaginary $\longleftrightarrow$ semisimplicity of flat bundles

Don't know good Higgs analog of regular singularities of $\mathcal{D}$-modules!

There's a $\mathbb{C}^*$ action on local systems (defined via nonabelian Hodge theory) & the fixed points are those that underly VHS .... so the idea is all local systems (semisimple) underly twistor structures & $\mathbb{C}^*$ equivariant ones underly VHS

Twistor $\mathcal{D}$-modules:

Look at the Rees algebra R of $\mathcal{D}$ over $X \times \mathbb{C}_\lambda$.

Consider $(\mathcal{M}', \mathcal{M}'', C)$ Rees modules

$$C : \mathcal{M}'|_{X \times \mathbb{C}^*_\lambda} \otimes \sigma^* \mathcal{M}''|_{X \times \mathbb{C}^*_\lambda} \longrightarrow \mathrm{Dist}^{\lambda\text{-hol}}_{X \times \mathbb{C}^*_\lambda}$$

(σ = antipodal on $\mathbb{P}^1$) distributions

where C is a $R \otimes \sigma^* R$-morphism.

Case $X = pt$, $R = \mathcal{O}_{\mathbb{C}_\lambda} \Longrightarrow$

$V/\mathbb{P}^1$, V_0, V_∞ fibers

$\Rightarrow$ triple $(V_0^\vee, \sigma^* V_\infty, C)$

$$C : V_0^\vee|_{\mathbb{C}^*} \otimes \sigma^*(\sigma^* V_\infty|_{\mathbb{C}^*}) \longrightarrow \mathcal{O}_{\mathbb{C}^*}$$

natural pairing

Consider nearby & vanishing cycles functors:
want to extend from $\mathcal{D}$-mod to polarized R-modules
Consider V-filtration

$$f \in V_a \mathcal{M}', \quad g \in V_b \mathcal{M}'' \qquad X_0 \times \mathbb{C}_t$$

$$\Rightarrow [f] \in \psi_* \mathcal{M}', \quad [g] \in \psi_*(\mathcal{M}'') \qquad \downarrow t$$

$$\mathbb{C}_t$$

$$\langle \psi_t(C)([f], \sigma^*[g]), \varphi \rangle \qquad (\varphi \text{ test fn on } X_0 \times \Delta_1)$$

$$\shortparallel$$

$$\operatorname{Res}_{s=-1} \langle C(f, \sigma^* g), \underset{\text{bump fn}}{\chi(t)} \varphi |t|^{2s} \frac{dt\, d\bar{t}}{2\pi i} \rangle$$

$\vdots$

.... define for $\operatorname{Re} s \gg 0$, analytically continue

<u>Def</u> Pure twistor D-module:

- $\mathrm{Supp}(\mathcal{M}', \mathcal{M}'', C) = \{p\}$
  $\Rightarrow (\mathcal{M}', \mathcal{M}'', C) = i_{p*}$ (pure twistor struct)

- $\mathrm{Supp}(\mathcal{M}', \mathcal{M}'', C)$ has $\dim > 0$
  $\Rightarrow$ take f function where $\mathrm{Supp} \subsetneq f^{-1}(0)$

  & take $\left.\begin{array}{l} \mathrm{Gr}^W \psi_f(\mathcal{M}', \mathcal{M}'', C), \\ \mathrm{Gr}^W \phi_f(\mathcal{M}', \mathcal{M}'', C) \end{array}\right\}$ should be pure twistor D-modules with appropriate weight.

The polarization is used to (inductively) force relation between $\mathcal{M}', \mathcal{M}''$.

$$\text{Pure twistor D-mod} \xrightarrow[\substack{\text{natural functor} \\ \text{set } \lambda = 0}]{} \mathrm{Coh}(T^*X)$$

But don't understand the image!