Graeme Segal - Topological Field Theory & Branes

Note Title 1/11/2007

Σ compact Riemann surface.

Two opposite ways to describe hol. line bundles

- global / smooth way: flat unitary connections
  $\longleftrightarrow$ Hom$(\pi_1(\Sigma), \mathbb{T})$

- local description: trivialize off fin many pts, get local twistings, labelled by $S^1 \longrightarrow \mathbb{T} \Longleftrightarrow \mathbb{Z}$.
  $\Rightarrow \mathrm{Alb}(\Sigma) \leftarrow \mathbb{Z}[\Sigma]$ universal abelian group generated by curve.

Nonabelian case: $[\pi_1(G) = 1]$

- global: generic bundles have flat unitary connections
- local: trivialize off fin many points, get local twisting data:
  locally can think we're on S^2.
  Bundles on $S^2 \longleftrightarrow \{\mathbb{T} \longrightarrow G_{cpt}\}/\sim$
  homomorphisms
  (analog of $\mathbb{Z} = \{S^1 \longrightarrow \mathbb{T}\}$ in abelian case)

Such homs$/\sim$ $\Longleftrightarrow$ irreducible representations of ^LG

But as <u>space</u> look very different: not <u>discrete</u>

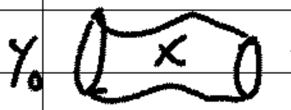
but rather connected geometry: eg trivial bundle is dense in bundles on S^2 !
[dense point in funny space of bundles...]

$\longrightarrow$ noncommutative space.
approach to these from TFT & D-branes!

Connes: bad spaces $\rightsquigarrow$ NC rings/Morita equiv
$\Longleftrightarrow$ category of modules.
So instead of spaces look at categories
In string theory there's emerged an intermediate range between commutative & NC rings ---- 2D QFTs.

d-dim QFT, schematically:

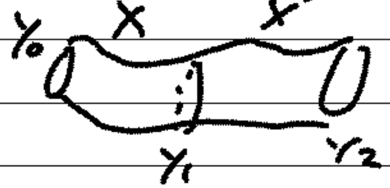
Y^{d-1} compact oriented (+ extra structure) $\longmapsto$ $\mathcal{H}_Y$ topological vector space



cobordism $Y_0 \overset{X}{\rightsquigarrow} Y_1$

$\longmapsto$ trace class operator
$$U_X : \mathcal{H}_{Y_0} \longrightarrow \mathcal{H}_{Y_1}$$

Axioms: 1. Concatenation:



$U_{X'} \circ U_X = U_{X' \cup X}$

2. tensoring: $\mathcal{H}_{Y_0 \amalg Y_1} \cong \mathcal{H}_{Y_0} \otimes \mathcal{H}_{Y_1}$.

4d theory $\Longrightarrow$ dimensional reduction: X^2 compact surface

$\times X^2$: 2-mflds $\longrightarrow$ 4-mflds
1-mflds $\longrightarrow$ 3-mflds

So get 2d TFT for every X^2.

2d TFT $\Longleftrightarrow$ commutative Frobenius algebra

$\mathcal{H}_{S^1} = A$


trace θ

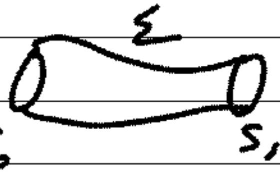

unit


product

Now lift to take values in differential graded vector spaces: grading will come from rotation of the circle, once we weaken axioms a little.

$S^1 \longmapsto C^\bullet_{S^1}$ cochain complex

$C^\bullet_{S^1 \amalg S^1} = C^\bullet_{S^1} \otimes C^\bullet_{S^1}$,



$\Rightarrow$ map $C^\bullet_{S_0} \xrightarrow[U_{\Sigma,\sigma}]{} C^\bullet_{S_1}$

Σ equipped with some extra structure σ, such as conformal structure or metric

$\Rightarrow U_\Sigma : C^\bullet_{S_0} \longrightarrow \Omega^\bullet(\text{Structures}(\Sigma); C^\bullet_{S^1})$

cochain map ..... family of cochain map labelled by structures on Σ.

M compact manifold : we'll replace $C^\infty(M)$ by $C^\infty(\mathcal{L}M)$, has interesting noncommutative multiplications

$$U_\Sigma : C^\infty(\mathcal{L}M) \times C^\infty(\mathcal{L}M) \longrightarrow C^\infty(\mathcal{L}M)$$

$$F_1 \qquad F_2 \longmapsto F_1 \underset{\Sigma}{*} F_2$$

Σ =  with conformal structure

$F_1 \underset{\Sigma}{*} F_2 \ (\gamma)$:  $\gamma \in \mathcal{L}M$

look at all maps of Σ with outside loop going to γ

$$F_1 \underset{\Sigma}{*} F_2(\gamma) = \int_{\varphi:\Sigma\to M} F_1(\gamma_1)\, F_2(\gamma_2)\, e^{-S(\varphi)}\, \mathcal{D}\varphi$$

Spread out multiplication (convolution over S^1) using an action / measure $e^{-S(\varphi)}\mathcal{D}\varphi$

Example: T-duality T compact torus, T^* dual torus

$$V/\Pi \qquad\qquad V^*/\Pi^*$$

$$\mathcal{L}T \times \mathcal{L}T^* \longrightarrow \mathbb{T}$$ perfect pairing, making these Pontryagin dual

$$\begin{array}{ccccccc}
\mathcal{L}T & = & T & \times & \pi_1(T) & \times & (\text{Vector space}) \\
 & & \updownarrow & & \updownarrow & & \updownarrow \\
\mathcal{L}T^* & = & \pi_1(T^*) & \times & T^* & \times & (\text{Vector space})^*
\end{array}$$

$\Longrightarrow$ so expect a Fourier transform

$$C^\infty(\mathcal{L}T) \simeq C^\infty(\mathcal{L}T^*)$$

exchanging multiplication & convolution

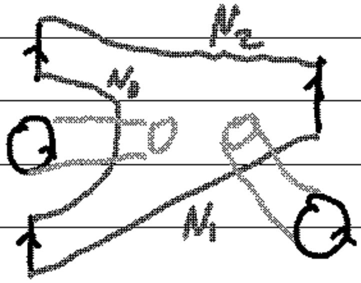
.... but in fact get isomorphism of σ-model algebra above on both sides!!

In limit of small $T \leftrightarrow$ big T^* see the algebras $C^\infty(T)$ or $C^\infty(T^*)$ which are very different as rings, but above σ-model algebra interpolates between them.

What are modules for chain complexes like $\Omega^\cdot(LM)$ with above NC (σ-model) multiplication?

D-branes (Polchinski):
allow open & closed strings. ask open strings to end on submanifolds N_0, N_1

So now get chain complexes $C^\bullet_{I, N_0, N_1}$ I = interval
giving open & closed 2d TFT

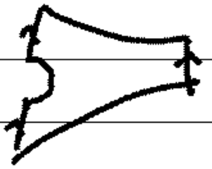


Cobordism of manifolds with bdry: give cobordism of the boundaries & ingoing & outgoing manifolds together as a boundary

$S_0 \mapsto C^\bullet_{S_0}$

Given a closed string theory ask for the maximal extension to an open-closed theory

→ form a category in a strict TFT due to picture



$$\mathcal{H}_{I,N_0,N_1} \otimes \mathcal{H}_{I,N_1,N_2} \longrightarrow \mathcal{H}_{I,N_0,N_2}$$

Now promote this to chain complexes

$\Rightarrow$ A_∞ category

Example: M symplectic, $LM \rightsquigarrow$ theory which produces the Fukaya category (A-model).

Case $M = T^*N$ (noncompact version of TFT)
$\rightsquigarrow$ can describe (part of) this as $D_c(N)$ constructible derived category.

Obvious generators: $Z \subset N$ submanifold,
$T^*_Z N \subset T^*N$ conormal (Lagrangian)
make a category out of these (following Sullivan):

$$C^\bullet_{I, Z_0, Z_1} = \Omega^\bullet(\mathcal{P}_{Z_0, Z_1}(N))$$

paths in N from Z_0 to Z_1.

Can twist by coefficient systems on Z_0, Z_1.

This satisfies all the conditions of TFT

$\rightsquigarrow$ Sullivan topological string theory category.

Put Morse fn on $\mathcal{P}_{Z_0, Z_1}$: critical points are intersection pts of submanifolds.....

N=2 SUSY 2d CFT:

look at Riemann surfaces equipped with Spin^c structure: pair of line bundles L_1, L_2 + $L_1 \otimes L_2 \xrightarrow{\sim} T\Sigma$.

(Spin structure: case $L_1 = L_2$)

N=2 SUSY: $\mathcal{H}_{S^1, L_1, L_2}$ carrying action of multiplication by sections of L_1, L_2

L_1, L_2 line bundles on S^1 $\rightsquigarrow$ odd part of N=2 super Virasoro algebra

Can specialize this in different ways:
eg set $L_1 = T\Sigma$, $L_2 = \mathbb{C}$
or in other order $\Longrightarrow$ get <u>two twists</u> of this theory: get differential from the twisted [illegible]

Structures on manifolds: can eg consider QFTs defined on manifolds equipped with maps to some target M (eg $BG \Longrightarrow$ gauge theory)

eg $d=1$: look at points + maps to M, cobordisms: vector bundles w/ connection on M (parallel transport along 1-dim cobordisms)

$d=2$ theories over $M \rightsquigarrow$ elliptic objects

Elliptic Objects

K-theory ⇝ vector bundles w/ connection over a space

$\wr$

certain 1-dim field theories

Consider d-1 manifolds Y^{d-1} + maps $Y^{d-1} \longrightarrow M$

Cobordism $Y_0 \overset{X}{\sim} Y_1$ all mapping to M

$\Rightarrow$ maps $\mathcal{H}_{Y_0 \to M} \longrightarrow \mathcal{H}_{Y_1 \to M}$

d=1 : get vector spaces parametrized by pts of M, & operators (parallel transport) for paths,

d=2 : get vector bundle on LM
+ string connection : given surface in M giving cobordism between a bunch of loops ⇒ get operators. eg cylinders give connection on LM but have many more operators.

Abelian case: all our vector bundles 1-dimensional

Deligne / Cheeger-Simons / differential cohomology:

$$\begin{cases} H^0_D(M) = H^0(M;\mathbb{Z}) \\ H^1_D(M) = C^\infty(M;\mathbb{T}) \end{cases}$$

$H^2_D(M)$ = line bundles w/ conn on M

$H^3_D(M)$ = gerbes ....

all topological abelian groups.

What are good local representatives?

$H^3_D(M)$: family of categories, locally torsors for $\{\mathbb{C}\text{-lines}\}$, + connection data

Theory $\mathcal{H}_{Y\to M}$ where all vector spaces are lines represents an element of $H^3_D(M)$.

In general however these objects don't appear local wrt M.

$H^*(M;\mathbb{Z})$, $K(M)$ classes are not local but know how to give local models: $K(M) \rightsquigarrow$ vector bundles, which do localize.

$H^*(M;\mathbb{Z})$: to promote to local objects, can consider category w/ objects $Z^p(M)$, & morphisms $\hom(Z_0, Z_1) = \{c \in C^{p-1} : dc = Z_1 - Z_0\}$

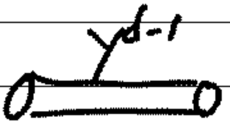
This category *depends* on the chain model. Get independent category if take hom to be above mod $d(C^{p-2})$, but now morphisms are not local.

To achieve locality must consider full p-category

2d field theories valued in M : bundles / $\mathcal{L}M$ with "string connection" — not local in M ... $\mathcal{L}M$ itself is not local in M.

Locality in Y : could try to cut Y into pieces $\Rightarrow$ pass to 3-tiered theories

$Z^{d-2} \longmapsto$ additive category $\mathcal{C}_Z$



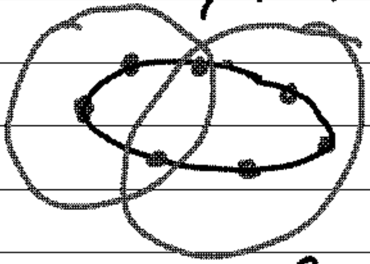
$\longmapsto$ functor $\mathcal{H}_Y : \mathcal{C}_{Z_0} \longrightarrow \mathcal{C}_{Z_1}$

$U_X : \mathcal{H}_{Y_0} \longrightarrow \mathcal{H}_{Y_1}$ natural transformation

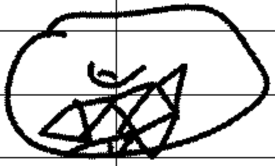
e.g A_Z algebra, $\mathcal{C}_Z = A_Z$-mod
functors $\mathcal{H}_Y$ are (A_{Z_0}, A_{Z_1})-bimod's etc.

Get gerbes on M (bundle of categories)
by considering such 3-tiered theories valued in M

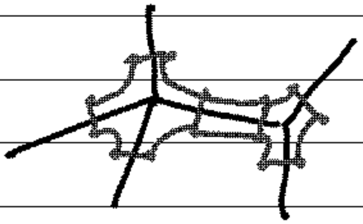
Get locality in M now: given loop
can cut it into small intervals
which are now local



2d theory ⇒ get numbers
for surfaces valued in M.
How to construct this? triangulate our surface
--- to points get categories, paths ⇝ functors etc



Thicken up

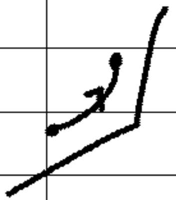


Red vertices ⟷ <u>flags</u> in the triangulation:
vertex, 1-simplex & 2-simplices together
⇒ get orientation: incoming & outgoing

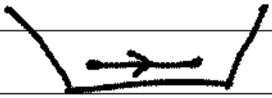
Have category $\mathcal{C}$ attached to each incoming points & to outgoing points
S_+ S_-

Have 3 cobordisms from S_+ to S_-:

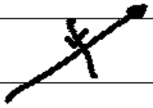
those corresponding to fixing vertex & 2 simplex



those corresponding to fixing 1- & 2-simplices



& those corresp to fixing 0- & 1-simplices



$$\Rightarrow 3 \text{ functors } \begin{tikzcd} \mathcal{C}^{\otimes S_+} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C}^{\otimes S_-} \end{tikzcd}$$

Now have "amoeba" cobordisms between the

,

&

.

All of these pieces are completely local, in nbhd of simplex.

$$\begin{tikzcd} \mathcal{C}^{\otimes S_+} \arrow[r, shift left=4, ""{name=A}] \arrow[r, ""{name=B}] \arrow[r, shift right=4, ""{name=C}] & \mathcal{C}^{\otimes S_-} \arrow[from=A, to=B, Rightarrow, shorten=1pt] \arrow[from=B, to=C, Rightarrow, shorten=1pt] \arrow[from=C, to=A, Rightarrow, bend right, shorten=1pt] \end{tikzcd}$$

— version of picture of Kontsevich to prove modular functors $\longleftrightarrow$ 3d TFT.

In elliptic cohomology we are supposed to have a forgetful functor $Ell^*(X) \longrightarrow K^*(X)[[t]]$

X orientable in suitable sense $\Longrightarrow$ elliptic genus, integral modular form.

Categorify: (in analogy with $\hat{A}$-genus = index($\not{D}$))

Witten: look at index($\not{D}$), get character of a graded vector space using rotation of loops. So first approx is a positively graded vector bundle, ie elt of $K^*(X)[[t]]$

K-Oriented X $\Rightarrow$ $1 \in K^0(X) \longrightarrow \underset{\hat{A}\text{ genus}}{\underset{\cup}{K^{-n}(pt)}}$

Candidate for an elliptic object on any manifold: free fermions on the tangent bundle

$\gamma \in \mathcal{L}M$ $\Gamma(\gamma^* TM)$ is polarized vector space (+ & − energy)

$\Rightarrow \mathcal{H}_\gamma = \Lambda^{\frac{\infty}{2}} \Gamma(\gamma^* TM)$ Fermionic Fock space, know how to propagate this along R.