Zhu p-adic CFT

A good conformal field theory Π has p-adic completions Π_p s.t. global theory $\Pi_{\mathbb{A}}$ is automorphic.

F local field: $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$ ...

$X = F((t)) \times F((t))dt \times F$ Heisenberg group

$a(t)\cdot b(t)dt = (b(t)dt)\, a(t)\, \mathrm{Res}\,(a(t)b(t)dt)$

X acts on formal functions $\mathrm{Fun}(F((t))) = \mathrm{Fun}(\ldots a_{-1}, a_0, a_1, \ldots)$ (to be defined)

$(a(t)\cdot f)(x(t)) = f(x(t)+a(t))$ translation

$(b(t)dt \cdot f)(x(t)) = \psi(\mathrm{Res}\; x(t)\, b(t)\, dt)\, f(x(t))$

$\psi : F \to \mathbb{C}^*$ additive character.

C curve$/F$,  $p \in C$ t local parameter

Functional $\phi : \mathrm{Fun}\, F((t)) \longrightarrow \mathbb{C}$ [conformal block?]

s.t. $\phi(g\cdot f) = \phi(f)$ for $g \in \Gamma(C \setminus p, \mathcal{O})$ or $g \in \Gamma(C\setminus p, \omega)$

-- subgroups of X at p.

$\phi(f)$ "=" $\sum_{x(t)\in F(C-p,\mathcal{O})} f(x(t))$ doesn't make sense ...

Set $D(F((t))) :=$ Span of $\bigotimes_{i=-\infty}^{\infty} f_i$

where f_i is of following types

1. $f_i = \psi_b(a_i) = \psi(b\cdot a_i)$
0. f_i Schwartz function of a_i
-1. $f_i = \delta_b(a_i) = \begin{cases} 1 & a_i = b \\ 0 & a_i \neq b \end{cases}$

$\Bigg\}$ allowable distributions

require condition $i << 0 \quad f_i = \delta_0$
$i >> 0 \quad f_i = \psi_0 \equiv 1$

-- finite linear span of such functions
in "smooth" part of continuous dual

i.e. part of $\varprojlim S(t^{-m} F[[t]] / t^n F[[t]])$

or ~~[illegible]~~ Gaitsgory-Kazhdan dual space

$\phi(f)$ should be element of completion of $(D(F((t)))^*_2$

X acts on $D(F((t)))$, & $\phi(f)$ makes sense on how

- summation on sections of line bundles with poles.

~~Our~~ $f_{-i} \equiv \delta_0$ for $i \leq -N \Rightarrow$

~~$\phi(f) = \sum_{x(f)}$~~

Theorem: There is a unique (up to scalar) continuous $\phi : D(F((t))) \longrightarrow \mathbb{C}$ which is invariant under $\Gamma(C-p, \mathcal{O}) \times \Gamma(C-p, \omega)$ ... Lagrangian subgroup (maximal abelian).

For C curve over $\mathbb{Q} \Rightarrow C_v$ curve over every completion $\mathbb{Q}_v$.

Def The space of multiplicative states $(\subset D(\mathbb{Q}((t))))$

~~[illegible]~~ $\mathcal{G}_v = \{ \text{span} \bigotimes_{-\infty}^{\infty} f_i \in D(\mathbb{Q}_v((t))) : f_i \text{ is either type } (1) \text{ or type } (-1) \}$

Global multiplicative states: $\mathcal{G}(\mathbb{Q}) \hookrightarrow \mathcal{G}_v$ all v

Theorem ~~[illegible]~~ Suppose C curve over $\mathbb{Q}$, $p \in C(\mathbb{Q})$, t rational local parameter $\Rightarrow$

~~[illegible]~~ $\phi_p(f)$ ... i.e. $\Longrightarrow$

Then $\Phi_p(v)$ is either 0 for all p, or
$\prod_{p_i = \text{places of } \mathbb{Q}} \phi_{p_i}(v) = 1$. [Note: we've fixed global additive character, $\prod \psi_{p_i} = 1$]

[why? $\phi_p(a_i, b_i) = \psi_p(\sum a_i b_i) = 1$]
quadratic form ~~steep~~

Picture: arithmetic curve $C \xrightarrow{\pi} \overline{\operatorname{Spec}\mathbb{Z}}$, 

for each $v \in \overline{\operatorname{Spec}\mathbb{Z}}$ have conformal field theory on p-adic curve = neighborhood of fiber of $\pi^{-1}(v)$

CFT here: $V = D(F((t)))$

For every $x \in C_v$, $D(F((t)))$ is an automorphic representation of Heisenberg X wrt subgroup

$$\left(\Gamma(C_v \setminus x, \mathcal{O}) \times \Gamma(C_v \setminus x, \omega) \times F\right) \rtimes SL_2\begin{pmatrix} \mathcal{O} & \omega \\ \omega^{-1} & \mathcal{O}\end{pmatrix}$$

Oscillator rep

For a horizontal divisor $C \underset{s}{\overset{\pi}{\rightleftarrows}} \overline{\operatorname{Spec}\mathbb{Z}}$ $\overline{C_S}$ = set of places of $s(\overline{\operatorname{Spec}\mathbb{Z}})$ ... i.e. residue field is a number field

Q: What are automorphic forms on such horizontal divisors?

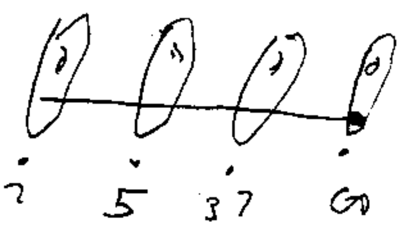
Want automorphic functional ϕ on $\mathbb{R}((t)) \times \mathbb{R}((t))dt \times \mathbb{R} \rtimes \widehat{SL_2}\,\mathbb{R}((t))$

wrt $\Gamma = \mathbb{Z}((t)) \times \mathbb{Z}((t))dt \rtimes \widehat{SL_2}\,\mathbb{Z}((t))$

Vertical picture: look at p curves over finite or local field

horizontal picture: look at $\mathbb{Q}$-points of the curve

– we fix point at arithmetic infinity



Schwartz space

X acts on $S(\mathbb{R}[t^{-1}]t^{-1} + \mathbb{R}[t^{-1}]t^{-1}dt) =: V$

$\circlearrowleft$

$\mathbb{R}[t^{-1}]t^{-1} \times \mathbb{R}[t^{-1}]t^{-1}dt$ acts as translation

~~translation~~ $\mathbb{R}[t] \times \mathbb{R}[t]\, dt$ acts as additive character

& can lift to semidirect product with SL_2

Consider half space $\tilde{H} = \{ u + vi \mid u, v \in \mathrm{End}_{\mathbb{R}}(V \oplus V^*)$ s.t. u, v define a symmetric bilinear form on V & $> 0 \}$

$L = \mathbb{Z}[t^{-1}]t^{-1} + \mathbb{Z}[t^{-1}]t^{-1}dt$

Would like to write an L-invariant function, using $\tilde{H}$.

... ie can only write functional on smaller space than full space, where things make sense

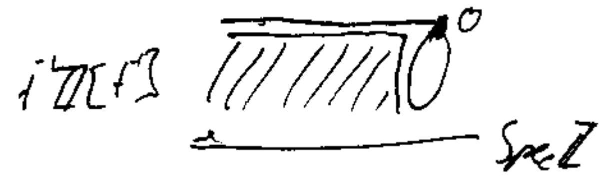
$\Omega \in \tilde{H}$ $\quad f_\Omega(v) = e^{\pi i (v, \Omega v)} \in S(V)$

$\phi(f_\Omega) = \sum_{h \in L} e^{\pi i (h, \Omega h)}$ diverges

... so restrict to smaller subspace ... roughly for each pt in $\tilde{H}$ get automorphic form.

$SL_2\, \mathbb{R}((t)) \circlearrowright \tilde{H}$ $\quad$ [each f_Ω gives Schwartz functional, on subspace generated by some f_Ω's will get functional]

$SL_2\, \mathbb{R}((t)) \hookrightarrow V \oplus V^*$



Let $H = Q \cdot Vir \cdot SL_2(\mathbb{R}((t))) \cdot iI$

$$Vir = \{ \sum_{i \geq 1}^{\infty} a_i t^i, \quad a_1 = 1 \}$$ group of change of parameters

$$Q = \{ q \mid 0 < q < 1 \}$$

$iI \in \tilde{H}$ s.t. I defines the standard inner product on V