$\boxed{SL_2}$ $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc=1 \right\}$, $a,b,c,d \in k$ field

- the fundamental object in Lie theory : $k = \mathbb{C}, \mathbb{R}, \mathbb{F}_p, \mathbb{Q}_p$
  first simple group building block of others,
- number theory : modular forms $SL_2(\mathbb{R}, \mathbb{Q}_p)$
- hyperbolic geometry : $SL_2\mathbb{R} = \mathrm{Isom}(H^2)$ $SL_2\mathbb{C} = \mathrm{Isom}(H^3)$
- algebraic geometry : Lefschetz $SL_2 \circlearrowright H^*(X)$ compact Kähler / $\mathbb{C}$ projective
- analysis : behind classical harmonic analysis on $\mathbb{R}^n$,
  Huyghens principle, ...
- physics : Lorentz group, particles are representations

<u>Representations</u>

G group, V vector space $/\mathbb{C}$

$\rho : G \longrightarrow \mathrm{Aut}\, V$ homomorphism : $g, v \longmapsto g \cdot v = \rho(g) v$.

(V, ρ) (W, ρ') representations $\rightsquigarrow$ a map of representations (intertwiner) is $\varphi : V \longrightarrow W$ $\mathrm{Hom}_G(V, W)$

$$g \cdot \varphi(v) = \varphi(g \cdot v)$$

G topological, V topological (eg fin dim)

$\rightsquigarrow$ ask $G \longrightarrow \mathrm{Aut}\, V$ continuous

— i.e. coefficients of matrices attached to group elements are continuous

$v \in V$ $v^* \in V^*$ [cont.] linear functional

$\rightsquigarrow$ f_{v,v^*} function on G "matrix element"

$$f_{v,v^*}(g) = \langle g \cdot v, v^* \rangle = v^*(g \cdot v)$$

$\rightsquigarrow$ map $V \otimes V^* \longrightarrow \text{Functions}(G)$

$v \otimes v^* \longmapsto f_{v,v^*}$ bilinear map

- specify types of representations by types of functions on G we get as matrix elements.

(e_i standard basis of $\mathbb{C}^n$ $f_{e_i, e_j^*}(g) =$ j^{th} component of $g \cdot e_i$ = i,j entry of matrix $\rho(g)$).

- V, W reps $\rightsquigarrow$ so is $V \oplus W$ $\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$
- V is <u>irreducible</u> if no nontrivial subrepresentations: no G-invariant subspaces
- V is <u>indecomposable</u> if can't write $V = W_1 \oplus W_2$, W_i G-representations

Example: · $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \simeq \mathbb{C} \subset GL_2\mathbb{C} = \text{Aut}\,\mathbb{C}^2$ (or $\mathbb{R}$)

$V = \begin{pmatrix} * \\ 0 \end{pmatrix} \subset \mathbb{C}^2$ has no U invariant complement: indecomposable but not irreducible

· $\mathbb{C}^2 \circlearrowleft SL_2\mathbb{C}$ irreducible: can get $g \cdot v =$ any given w if $v, w \neq 0$.

$$\begin{array}{ccc} \text{compact abelian} & \subset & \text{compact} \\ \cap & & \cap \\ \text{abelian} & \subset & \text{Lie groups, eg } SL_2 \end{array}$$

compact: indecomposables = irreducibles

abelian: ireds 1-dimensional

G finite : finite dimensional representations are semisimple/ completely decomposable: every $V \cong V_1 \oplus \ldots \oplus V_k$ sum of irreducibles [so irred $\Leftrightarrow$ indecomp]
same holds for G compact.

Why?

Def (V, ρ) is a unitarizable representation if $\exists \langle , \rangle$ Hermitian inner product on V which is G-invariant : $g\langle , \rangle (:= \langle g-, g- \rangle) = \langle , \rangle$
$(V, \rho, \langle , \rangle)$ is a unitary representation.

Prop A unitarizable rep (V, ρ) is semisimple

Pf $W \subset V$ subrep $\Rightarrow$ so is $W^\perp$ for $\langle , \rangle$ G-invt
$\rightsquigarrow V = W \oplus W^\perp$ every sub has a complement. $\blacksquare$

Prop G finite $\rightsquigarrow$ any rep V is unitarizable

Pf Take $\langle , \rangle_0$ any Hermitian inner product
Set $\langle , \rangle = av(\langle , \rangle_0) = \frac{1}{|G|} \sum_{g \in G} g \langle , \rangle_0$

$\rightsquigarrow \langle , \rangle$ G-invariant, still nondegenerate :
$\|v\|^2 = \langle v, v \rangle = \frac{1}{|G|} \|g \cdot v\|^2 > 0 \quad v \neq 0.$ $\blacksquare$

More generally same holds for G compact : $\frac{1}{|G|} \sum_G \rightsquigarrow \int_G d\mu$
$d\mu$ Haar measure on G, eg $\frac{d\theta}{2\pi}$ on $U(1) = S^1$:
left invariant integration $C(G, \mathbb{R}) \xrightarrow[\int d\mu]{} \mathbb{R}$
- take volume form $dx_1 \wedge \ldots \wedge dx_n$ at one point 1, define at all $g \in G$ by translation, normalize to have volume 1.

ev is injective: $\mathrm{Ker}(ev) \subset W_{\varphi_1} \oplus \dots \oplus W_{\varphi_k}$ $\varphi_1, \dots, \varphi_k$ lin. indep.

G-rep (must project to all or none of each factor)

for any G & V

$\varphi_k(w_0)$ in span of $\varphi_1, \dots, \varphi_{k-1}$ $\Rightarrow$ $\varphi_k(w_0) = \sum \lambda_i \varphi_i(w_i)$

$\varphi_k(g w_0) = \sum \lambda_i \varphi_i(g w_i)$ $\quad$ $w_0 \mapsto w_i$ gives

$\rightsquigarrow$ pass to $\mathrm{span}(G \cdot w_0) = W$ $\rightsquigarrow = \sum \lambda_i' \varphi_i(g w_0)$ $\quad$ isom $W \to W$ i.e. scalar $w_i = \mu_i w_0$

φ_k is linear combo of φ_i

contradiction ∎

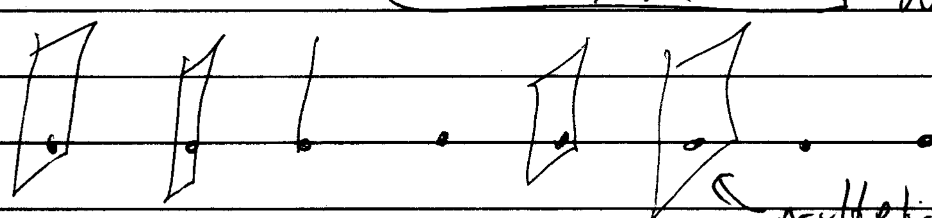
So $\mathrm{Hom}_G(W,V) \otimes W \hookrightarrow V$ always.

If we know V is semisimple (eg G compact)

$[V \cong W_0 \oplus \dots \oplus W_0 \oplus W_1 \oplus \dots \oplus W_1 \oplus \dots \oplus W_k \oplus W_k \dots \oplus W_k]$

$\rightsquigarrow$ know that these maps also span

$$V = \bigoplus \underbrace{\mathrm{Hom}_G(W,V)}_{\text{multiplicity space}} \otimes W \quad (W \text{ isotypic component})$$



list of irreps

multiplicity spaces

Key questions:

- Construct & describe irreps
- decompose specific representations of interest

Good place to start — most obvious representation, the regular representation: functions on G itself under (left) translation...

Fourier series $G = \mathbb{T} = S^1$ the circle group

$V = L^2(S^1)$ (left) regular rep

$\alpha \in \mathbb{T}$ $\quad f \in V$ $\quad (\alpha \cdot f)(\theta) = f(\theta - \alpha)$

pull back of f under $\theta \longmapsto \theta + \alpha$

— large collection of commuting operators on V,
try to simultaneously diagonalize $\longleftrightarrow$ decompose representation

$\rightsquigarrow$ exponentials are the characters of S^1:

$\chi : S^1 \longrightarrow \mathbb{C}^*$ homomorphism [nec. lands in $S^1 = U(1)$]

$\chi(\theta_1 + \theta_2) = \chi(\theta_1)\,\chi(\theta_2)$ possible eigenvalues

solved by $\chi_n(\theta) = e^{2\pi i n \theta}$

$$\alpha \cdot \chi_n(\theta) = e^{2\pi i n(\theta - \alpha)} = \chi_{-n}(\alpha)\,\chi_n(\theta)$$

$\rightsquigarrow \chi_n$ eigenvector with eigenvalue χ_{-n}.

$\rightsquigarrow$ χ_{-n} isotypic component $V_{\chi_{-n}} = \mathbb{C} \cdot \chi_n$.

How do we project a function to $V_{\chi_{-n}}$?

Can use L^2 inner product $f \longmapsto \langle f, \chi_n \rangle \chi_n$

$$\langle f, \chi_n \rangle = \int f(\theta)\, e^{+2\pi i n \theta}\, d\theta =: \hat{f}(+n)$$
$$= \mathrm{av}\,(f \cdot \chi_{+n})$$

What is this representation theoretically?

V rep of S^1 $\rightsquigarrow$ $V \otimes \mathbb{C} e^{+2\pi i n \theta}$ new rep

$g \cdot v \otimes w = g \cdot v \otimes g \cdot w$

$$\begin{tikzcd}
\left[V \otimes \mathbb{C} e^{+2\pi i n \theta}\right]^{S^1} \arrow[r, phantom, "="] & V_{\chi_n} \\
V \otimes \mathbb{C}\chi_n \arrow[u, "\mathrm{av}"] & V \arrow[l, "\otimes \chi_n"'] \arrow[u, "\hat{f}(n) \cdot \chi_n"']
\end{tikzcd}$$

$\hat{\mathbb{Z}} = \mathbb{T}$: $\chi : \mathbb{Z} \to \mathbb{T} \longmapsto \chi(1) \in \mathbb{T}$.

(compact $\longleftrightarrow$ discrete)

reps completely decomposed   reps presented by generators, algebraically

$\hat{\mathbb{R}} \simeq \mathbb{R}$ $t \in \hat{\mathbb{R}} \longmapsto e^{2\pi i x t} = \chi_t(x)$ character.

V fin dim vector space $\hat{V} = V^*$ $v, v^* \longmapsto e^{2\pi i \langle v^*, v \rangle} \in \mathbb{T}$

... have a pairing $g \in G$, $\chi \in \hat{G}$ $\longmapsto \langle \chi, g \rangle = \chi(g) \in \mathbb{T}$

ie $G \times \hat{G} \to \mathbb{T}$ circle valued function on product

$\rightsquigarrow$ defines an integral transform

$f \longmapsto \hat{f}(\chi) = \int f(g) \langle \chi, g \rangle dg$

once Haar measures have been defined...

$g \in G$ acts on $\mathrm{Fun}(\hat{G})$ by multiplication operator ($G \to \mathrm{Fun}^*$):

g defines a function $g(\chi) = \langle \chi, g \rangle$ on $\hat{G}$

so can multiply by this function:

representation is already simultaneously diagonalized in basis of points of $\hat{G}$

Plancherel Theorem $L^2(G) \simeq L^2(\hat{G})$ as G reps

For G finite or compact, look at finite functions $L^2(G)^{fin}$

= finite linear combinations of characters

— all we're doing is writing functions in basis of eigenfunctions

Fourier transform — Pontryagin duality:

Goal: simultaneously diagonalize all $g \in G$ LCA group acting on $Fun(G)$ [or any other representation].

What does it mean to diagonalize a bunch of operators? $\mathcal{O}_1, \dots, \mathcal{O}_k \circlearrowright V$

Find a set X & a bunch of functions $f_1, \dots, f_k$ and assign $x \in V \rightsquigarrow V_x \subset V$

$V = \bigoplus_{x \in X} V$ so that $\mathcal{O}_i$ acting on V is multiplication by f_i: $v = \bigoplus_{x \in X} v_x$,

$\mathcal{O}_i \cdot v = \bigoplus_{x \in X} f_i v_x$

— X is joint spectrum of our operators,

$x: f_i \longmapsto f(x)$ eigenvalues, & V_x eigenspace for eigenvalue x.

[functions on a space always diagonalized in "basis" of δ-functions : $f \cdot \delta_x = f(x) \delta_x$].

$G = S^1$ V finite dimensional $\Rightarrow$ $V = \bigoplus_{n \in \mathbb{Z}} V_n$ $(X = \mathbb{Z})$

$\theta \cdot v_n = e^{2\pi i n \theta} v_n$.

ie $\theta \in S^1 \rightsquigarrow$ function $\{e^{2\pi i n \theta}\}$ on $\mathbb{Z} = \widehat{G}$.

$V = \text{functions}(S^1) \rightsquigarrow V = \widehat{\bigoplus_{n \in \mathbb{Z}} V_n}$ completed version

V_n is $\mathbb{C} \cdot z^n$

$\rightsquigarrow$

$V \simeq$ functions of some kind on $\mathbb{Z}$, & θ's act by multiplication.

G LCA, $\hat{G} = \mathrm{Hom}_{\text{top grp}}(G, U(1))$

$\rightsquigarrow g \in G$ gives function $\langle -, g\rangle$ on $\hat{G}$

$g : \chi \in \hat{G} \longmapsto \langle \chi, g\rangle = \chi(g)$

$g = hk \Rightarrow \langle -, g\rangle = \langle -, hk\rangle = \langle -, h\rangle\langle -, k\rangle$

$\langle -, g^{-1}\rangle = \{\chi \mapsto \chi(g^{-1}) = \chi^{-1}(g)\} = \langle -, g\rangle^{-1}$

$\Rightarrow$ Functions $(\hat{G})$ form a representation of G on which all g's simultaneously diagonalized

...... see if have $V = \bigoplus_{\chi \in \hat{G}} V_\chi$ with G acting by multiplication

Plancherel theorem: $\hat{}: L^2(G) \xrightarrow{\sim} L^2(\hat{G})$ isometry as G-reps

- ie Fourier transform "solves" $G \circlearrowright L^2(G)$.

$\hat{f}(\chi) = \int_G \overline{\chi(g)} f(g)\, dg$ = "projection of f onto vector $\chi(g)$" χ ....

Problem: χ often not in L^2 ... eg for $G = \mathbb{R}$!

Neither are δ-functions!

Nonetheless defining property of $(\ )^\wedge$ is

- Characters $\longleftrightarrow$ δ-functions

  δ-functions $\longleftrightarrow$ characters

  $\langle \chi, g\rangle$ is character of either argument,

gives $G \longrightarrow \hat{\hat{G}}$

Pontrjagin duality: $\langle, \rangle : G \xrightarrow{\sim} \hat{\hat{G}}$.

Case of $\mathbb{R}$: Schwartz space $\mathcal{S}(\mathbb{R}) = \{f\ C^\infty$ of rapid decay$\}$

$\delta(t), e^{2\pi i x t} \in \mathcal{S}(\mathbb{R})^* =$ tempered distributions x^n (all derivatives) $\to 0$

Theorem: $(\ )^\wedge : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}),\ \mathcal{S}(\mathbb{R})^* \xrightarrow{\sim} \mathcal{S}(\mathbb{R})^*$.

Spectral decomposition

V finite dimension, $T \in \mathrm{End}\, V$

$\Rightarrow V$ is $\mathbb{C}[x]$ module, $x^n \mapsto T^n$.

$\mathrm{Spec}\, \mathbb{C}[x] = \mathbb{C}$ affine line : max. ideals $\langle (x-\lambda) \rangle$

$$I \subset \mathbb{C}[x] \longrightarrow \mathrm{End}\, V$$
$$\twoheadrightarrow \mathbb{C}[T] \hookrightarrow$$

$\mathbb{C}[T] = \mathbb{C}[x]/I$ = polynomial functions on spectrum $\mathrm{Spec}\, T \subset \mathbb{C}$:

generalized eigenvalues = vanishing locus of minimal polynomials

$$\cancel{\mathbb{C}[T]} = \quad V = \bigoplus_i \mathbb{C}[x]/(x-\lambda_i)^{n_i}$$

$$T \leftrightarrow x \mapsto \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

Jordan blocks : nonsemisimple modules over $\mathbb{C}[x]$.

Relation to Fourier transform:

V as above $\Longleftrightarrow$ f.d. representations of $\mathbb{R}$:

$x \in \mathbb{R}$ acts as $e^{ixT} = Id + ixT + \frac{(ixT)^2}{2} + \frac{(ixT)^3}{3!} + \dots$

$iT = \frac{d}{dx} \rho(x) \big|_{x=0}$ $\qquad$ (T ~~self-adjoint~~ hermitian $\longrightarrow \rho$ unitary)

$\cancel{T + T^* = 0 \leftrightarrow \exp(T)\exp(T^*) = 1}$

Where do Jordan blocks come from?

$\mathbb{C}e^{isx}$, translation $\tau_y e^{isx} = e^{is(x-y)} = e^{-isy} e^{isx}$.

$\mathbb{C}\, x e^{isx}$: $\tau_y x e^{isx} = (x-y) e^{is(x-y)} =$

$$e^{-isy}(x e^{isx}) - y e^{-isy}(e^{isx})$$

$$\tau_y \begin{pmatrix} e^{isx} \\ x e^{isx} \end{pmatrix} = \begin{pmatrix} e^{-isy} & -y e^{-isy} \\ 0 & e^{-isy} \end{pmatrix} \begin{pmatrix} e^{isx} \\ x e^{isx} \end{pmatrix}$$

$$i \frac{d}{dy} \tau_y \big|_{y=0} = i \begin{pmatrix} -is & 1 \\ 0 & -is \end{pmatrix} = \begin{pmatrix} s & i \\ 0 & s \end{pmatrix}$$