$SL_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$

How to understand? as symm of spaces on which it acts

$SL_2\mathbb{C} \circlearrowright \mathbb{C}^2$. Transitive on $\mathbb{C}^2 \setminus 0$

$\Rightarrow \mathbb{C}^2 \setminus 0 = SL_2\mathbb{C}/N$, $N = \mathrm{Stab}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \simeq \mathbb{C}$

$SL_2\mathbb{C} \circlearrowright \mathbb{CP}^1 = \mathbb{C}^2 \setminus 0 / \mathbb{C}^*$: commutes with $\mathbb{C}^* \circlearrowright \mathbb{C}^2$.

$z = \frac{x}{y}$ coordinate on $\begin{pmatrix} x \\ y \end{pmatrix}$ for $y \neq 0$, ie complement of line $\begin{pmatrix} * \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$ Möbius transformations

Transitive (holomorphic) action. $H = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$\simeq \mathbb{C}^* \ltimes \mathbb{C}$

$\Rightarrow \mathbb{CP}^1 = SL_2\mathbb{C}/B$, $B = \mathrm{Stab}\begin{pmatrix} * \\ 0 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

(standard) Borel subgroup (stabilizer of line $\longleftrightarrow$ Borel).

$\pm I = Z(SL_2\mathbb{C})$ acts trivially $\rightsquigarrow$ action of $PSL_2\mathbb{C} = SL_2\mathbb{C}/\pm I$

[Another POV:

$PSL_2\mathbb{C} \simeq SO_{1,3}^+$ Lorentz group $\subset SL_4\mathbb{R}$

-- preserves $t^2 - x^2 - y^2 - z^2$ & direction of time

S^2 = sphere of positive light rays in $\mathbb{R}^4$

... $\|v\| = 0 / \mathbb{R}^+$ celestial sphere]

$\mathbb{CP}^1 = SL_2\mathbb{C} / B = (SU_2 B^>_-)/B = SU_2 / SU_2 \cap B$
$= SU_2 / T = SU(2)/U(1)$

$SU_2 = \{ A\bar{A}^t = Id \quad \det = 1 \} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$

= unit quaternions $t + xi + yj + zk$, $t^2 + x^2 + y^2 + z^2 = 1$
via $z + wj \longmapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$

$SU_2 \circlearrowright \mathbb{H} \cong \mathbb{R}^4 \supset \mathbb{R}^3 = \{xi + yj + zk\}$ via conjugation
acts orthogonally
$u \in \mathbb{R}^3$ unit $\Rightarrow$ $g = \cos\frac{\theta}{2} + u \sin\frac{\theta}{2}$
unit quaternion, acts on $\mathbb{R}^3$ as rotation
by θ around u.

$$\begin{array}{ccc} & SL_2\mathbb{C} & \\ \cup & & \cup \\ SU_2 & & SL_2\mathbb{R} \\ \cup & & \cup \\ SO(2) = U(1) & & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : |a|^2 + |b|^2 = 1 \end{array}$$

acts on $\mathbb{CP}^1$ w/ 3 orbits
$\mathbb{RP}^1 = SL_2\mathbb{R}/B_{\mathbb{R}}$, $\mathbb{H} = SL_2\mathbb{R}/U(1)$, $U(1) = \mathrm{Stab}(i)$

$S^2 = SU_2/U(1)$:
$S^1 \to S^3$
$\downarrow$
S^2 Hopf fibration

- SO_3: ball of radius π, ends identified $\cong \mathbb{RP}^3$
- Hairy ball thm $\Rightarrow$ every element of $SO(3)$ has fixed point on S^2
$\leadsto$ is rotation around an axis.
$\leadsto$ any ~~ele SU_2 or SO_3 is contained in a max~~

Theorem K compact connected $\Rightarrow$ any $k \in K$ is conjugate into a (fixed) maximal torus $T \simeq U(1)^n$

- in fact any connected abelian subgroup conjugate into T ($\Rightarrow$ any two max tori conjugate)

eg SU_2 any element diagonalizable
$\rightsquigarrow$ so $SU(2)$ made out of $U(1)$'s...
expect its rep theory to be made out of $U(1)$'s too:
ie out of Fourier series.

$SU_2 \curvearrowright V$ finite dimensional representations:

$\overset{\cup}{T}$ get rep of circle T

$$\Rightarrow \quad V = \bigoplus_{n \in \mathbb{Z}} V_n \quad \text{sum of } T\text{-reps}, \quad \underset{U(1)}{\overset{\cap}{z}} \mapsto z^n \text{ on } V_n.$$

What can we say of pattern of V_n?

- H group, $\alpha \in \mathrm{Aut}\, H$, V rep of H

$\rightsquigarrow$ new rep V^α: same vector space,

$\pi^\alpha(h) = \pi(\alpha(h))$

[Note $\pi^\alpha(h_1 h_2) = \pi(\alpha(h_1 h_2)) = \pi(\alpha(h_1)\alpha(h_2))$
$= \pi^\alpha(h_1)\,\pi^\alpha(h_2)$]

$H \subset G \rightsquigarrow N(H) \subset G$ acts on H by automorphisms
- in fact $N(H)/Z(H)$ acts.

$G \curvearrowright V = \bigoplus V_i$ H decomposition $\rightsquigarrow$

~~Symmetry between N(H)~~

Claim: Pattern of decomposition $\oplus V_i$ has a symmetry of $N(H)$.

W irrep of H, $i : W \hookrightarrow V$, $n \in N(H)$

$\Rightarrow$ $n i(w) = n \circ (i(w))$ gives an embedding of the n-twisted rep $W^n \hookrightarrow V_i$

$$h \cdot n i(w) = n\, h^{(n)} i(w) = n\, i(h^{(n)} w).$$

Our case: $H = T$ abelian, $W = N(T)/T \hookrightarrow T$ by conj.

$N(T)$: matrices that preserve decomposition $\mathbb{C}^2 = \mathbb{C}_1 \oplus \mathbb{C}_{-1}$ into eigenspaces $\Rightarrow$ monomials $N(T) = \underbrace{\begin{pmatrix} \circ & \\ & \circ \end{pmatrix}}_{T} \cup \begin{pmatrix} & \circ \\ \circ & \end{pmatrix}$

$W (= N(T)/T) \simeq \mathbb{Z}/2\, w$

$w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in N(T)$ order $\underline{\underline{4}}$ not 2 -

can't find an order 2 coset rep in $SU(2)$ [can in $U(2)$]

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$$

ie w acts on T by $a \mapsto a^{-1}$.

$\Rightarrow V = \oplus V_n$ $V_n \simeq V_{-n}$: weights are symmetric.

$\mathfrak{sl}_2$ vs SL_2

G Lie group $\xrightarrow{Lie}$ $\mathfrak{g} := T_1 G$ tangent at identity
$= Lie(G)$

1. Lie is a functor Lie groups $\longrightarrow$ Lie algebras: ie. $\mathfrak{g}$ has a Lie algebra structure in a canonical way & maps of Lie groups induce maps of Lie algebras. $\varphi: G \to H \Rightarrow Lie(\varphi): Lie(G) \to Lie(H)$.

$$\begin{array}{ccc} Lie(\varphi): & Lie(G) \longrightarrow & Lie(H) \\ \| & \| & \| \\ d\varphi|_1: & T_1(G) \longrightarrow & T_1(H) \end{array}$$

2. Lie's theorem: Lie is an equivalence of categories when restricted to 1-connected [:= connected, simply connected] groups.

Basic construction: 1-parameter subgroups: group homomorphisms $\{\psi: \mathbb{R} \to G\} \longmapsto d\psi|_1(1) \in \mathfrak{g}$, where $1 \in \mathbb{R} = T_1\mathbb{R}$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\psi"] \arrow[dr] & G \arrow[d, "\varphi"] \\ & H \end{tikzcd}$$

— functorial for Lie group homomorphisms

Theorem: this is a bijection $Lie(G) \leftrightarrow Hom_{grp}(\mathbb{R}, G)$.

If $G \overset{subgp}{\subset} GL_n\mathbb{R} \overset{open}{\subset} \mathbb{R}^{n^2}$ can realize this explicitly
$\mathfrak{g} \subset \mathfrak{gl}_n\mathbb{R} = Mat_n\mathbb{R}$ by exponential map.

exp: $M_n\mathbb{R} = \mathfrak{gl}_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$

$A \longmapsto e^A$ or e^{tA} 1-parameter subgroup $e^{t_1A + t_2A} = e^{t_1A}e^{t_2A}$

Differential invertible at 0: local diffeomorphism.

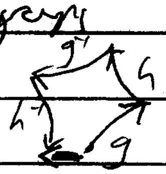
$\exp(tA)$ unique solution of ODE $f'(t) = Af(t)$
with $f(0) = Id$, so any 1-parameter subgroup
is of this form (take derivative of $\gamma(t_1)\gamma(t_2) = \gamma(t_1+t_2)$ rel t_2)

<u>Abstractly</u>: $T_1 G \simeq \{$left invariant vector fields on $G\}$
$g \mapsto \xi(g)$, $\xi(h \cdot g) = dh\, \xi(g)$
(trivialize TG under left translation action of G, $TG \simeq G \times \mathfrak{g}$).
So any $x \in \mathfrak{g}$ defines vector field l_x on G
$\rightsquigarrow$ ODE, have unique trajectory through ~~any~~ identity
– check it's a 1-parameter subgroup.
[$\Longrightarrow$ define $\exp: \mathfrak{g} \to G$ as value at 1 of $\varphi: \mathbb{R} \to G$.]

- <u>Lie algebra structure</u>:
  $x \mapsto l_x$ embeds $\mathfrak{g} \hookrightarrow Vect(G)$ Lie algebra
  Bracket on $Vect(G)$ coordinate independent $\Rightarrow$ preserved by diffeomorphisms, so condition of being left invariant preserved by bracket
  .... bracket of translations is translation of brackets.

Via exponential map: want to "linearize" group multiplication to operation on $T_1 G$:
$$\exp(tA)\exp(tB) = \exp\left(tA + tB + \tfrac{1}{2}t^2[A,B] + \dots\right)$$

Via 1-parameter subgroups: g_t, h_t 1-parameter subgroups
$\rightsquigarrow \frac{d}{dt} g_t h_t g_t^{-1} h_t^{-1}\big|_{t=0} \in T_1 G$



$\det : GL_n\mathbb{C}/\mathbb{R} \longrightarrow GL_1\,\mathbb{C}/\mathbb{R}$ Lie group map

$\Longleftrightarrow$ $\mathrm{tr} : \mathfrak{gl}_n\mathbb{C} \longrightarrow \mathbb{C}$ Lie algebra map

ie $\frac{d}{dt}(\det e^{tA})\Big|_{t=0} = \mathrm{tr}\,A$

So $\mathrm{Lie}(SL_n) = \mathfrak{sl}_n$

Similarly If $K = e^{tA}$ has $KK^t = 1$

$\Rightarrow A + A^t = 0$:

$\mathrm{Lie}(O_n) = \mathfrak{so}_n$ skew symmetric matrices

(automatically trace 0)

& $\mathrm{Lie}(U_n) = \mathfrak{u}_n$ $A + \bar{A}^t = 0$.

Lie's theorem also implies for G 1-connected

$$\left\{\begin{matrix}\text{f.d. reps of } \mathrm{Lie}\,G \\ \text{ie } \mathfrak{g} \to \mathfrak{gl}_n\mathbb{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{f.d. reps of } G \\ \text{ie } G \longrightarrow GL_n\mathbb{C}\end{matrix}\right\}$$

eg $G = SL_2\mathbb{C}$ 1-connected $(\sim S^3 \times \mathbb{R}^3)$

or $K = SU_2 \sim S^3$

- $\mathrm{Rep}\,SL_2\mathbb{C} \cong \mathrm{Rep}\,\mathfrak{sl}_2\mathbb{C}$ &
- $\mathrm{Rep}\,SU_2 \simeq \mathrm{Rep}\,\mathfrak{su}_2$: this one is concrete

since any element in SU_2 lives in a one-parameter subgroup, rotations around an axis

$x \in SU_2$ $\exp(tu) = \cos t + u \sin t$ $u \in \mathrm{Im}\,\mathbb{H}$, $u^2 = -1$

$\exp : \mathfrak{su}_2 \twoheadrightarrow SU_2$

[Theorem: $\exp : \mathfrak{g} \twoheadrightarrow G$ for any compact G]

Weyl's Unitary Trick f.d. Reps of $SL_2\mathbb{C}, SL_2\mathbb{R}, SU_2, sl_2\mathbb{C}, sl_2\mathbb{R}, su_2$ all identified !

Why? $\mathfrak{g}$ $\mathbb{R}$-Lie algebra, $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ complexification
$\Rightarrow$ Rep $\mathfrak{g}$ $\simeq$ Rep $\mathfrak{g}_\mathbb{C}$ because we consider complex reps:
any $\mathbb{R}$-linear map $\mathfrak{g} \longrightarrow \mathfrak{gl}_n\mathbb{C} = Mat_n\mathbb{C}$
extends uniquely by linearity to $\mathfrak{g}_\mathbb{C} \longrightarrow \mathfrak{gl}_n\mathbb{C}$

So Rep($su_2 = \mathbb{R}\langle \begin{pmatrix} i & \\ & -i\end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix}\rangle$)
= Rep ($sl_2\mathbb{C} = \mathbb{C}\cdot\langle \,''\, \rangle$)
= Rep $sl_2\mathbb{R}$!

$\Rightarrow$ Rep $sl_2\mathbb{C}$ $\simeq$ Rep SU_2 semisimple:
any representation is direct sum of irreducibles!

V f.d., $sl_2\mathbb{R} \curvearrowright V \Rightarrow sl_2\mathbb{C} \curvearrowright V \Rightarrow SL_2\mathbb{C} \curvearrowright V \Rightarrow SL_2\mathbb{R} \curvearrowright V$
[even though $SL_2\mathbb{R}$ is not simply connected!

$SL_2\mathbb{R} \curvearrowright \mathbb{CP}^1 = \mathbb{H} \cup \mathbb{RP}^1 \cup \mathbb{H}$
3-dimensional $\mathbb{H} \simeq SL_2\mathbb{R}/SO_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$]

$\Rightarrow$ Any f.d. rep of $SL_2\mathbb{C}$ (or any of the above)
is a direct sum of copies of reps V_n of $sl_2\mathbb{C}$
where V_n contains a highest weight vector of
weight $n \in \mathbb{N}$, $\dim V_n = n+1$.

What about SO_3? $= SU_2/\pm 1$ (or $PSL_2\mathbb{C} = SL_2\mathbb{C}/\pm Id$): only reps of
SU_2 where ± 1 act trivially. eg $V_1 = \mathbb{C}^2$ not, $V_2 = \mathbb{R}^3 \otimes \mathbb{C}$ is.
$e^{ith} = e^{it\begin{pmatrix} 1 & \\ & -1\end{pmatrix}} = \begin{pmatrix} e^{it} & \\ & e^{-it}\end{pmatrix} \Rightarrow t=\pi$ get $-Id$. Need all h-eigenvalues even $\Leftrightarrow n$ even !!

Construction of irreps of $SL_2\mathbb{C}$

$G \circlearrowright V, W$ vector spaces $\Rightarrow$ $G \circlearrowright V \otimes W$

$g \cdot (v \otimes w) = gv \otimes gw$

$\mathfrak{g} = \text{Lie } G \circlearrowright V \otimes W$: $x \cdot (v \otimes w) = xv \otimes w + v \otimes x \cdot w$

$= \frac{d}{dt}(e^{tx} v \otimes e^{tx} w)|_{t=0}$

[$\Rightarrow$ action on V^* : $e^{tx} \cdot v^*(v) = v^*(e^{-tx} v) \rightsquigarrow x \cdot v^*(v) = -v^*(x \cdot v)$]

V rep $\Rightarrow V^{\otimes n}$ rep. So is $\text{Sym}^n V$ n^{th} symmetric power.

- $\text{Sym}^n V \subset V^{\otimes n}$ invariants of permutations:

$\circlearrowleft S_n$ $\quad \sum a_{\vec{i}} v_{\vec{i}} \mapsto \sum a_{\vec{i}} v_{\sigma(\vec{i})}$ $\sigma \in S_n$

$\ldots x \otimes y \ldots \mapsto \ldots y \otimes x \ldots$

- Alternatively: $V^{\otimes n} \twoheadrightarrow \text{Sym}^n V = V^{\otimes n} / \ldots x \otimes y \ldots - \ldots y \otimes x \ldots$

coinvariants of S_n: identify expressions $x \otimes y$ & $y \otimes x$

[write $\sum a_{i} x^{n_i} y^{m_i} \ldots$ -- homogeneous polynomials of deg n]

- These two vector spaces are isomorphic via symmetrization $V^{\otimes n} \twoheadrightarrow$ symmetric subspace

$\underset{w}{\overset{\cup}{}} \mapsto \frac{1}{n!} \sum_{S_n} \sigma \cdot w.$

... project onto summand which is trivial S_n-isotypic component.

$GL(V) \circlearrowright V^{\otimes n} \circlearrowleft S_n$ commuting actions ($GL(V) \times S_n$ acts)

$\sigma(gv \otimes gw \otimes \ldots) = gw \otimes gv \otimes \ldots = g \cdot (\sigma(v \otimes w \ldots))$

So $GL(V)$ preserves S_n-invariants or projection to S_n-coinvariants.

$\Rightarrow G \to GL(V)$ acts on $V^{\otimes n}$ commuting with S_n

$\Rightarrow G \circlearrowright \text{Sym}^n V$.

Same holds for $\Lambda^n V = (V^{\otimes n})^{\text{sgn}}$ where sign is the ~~unique nontrivial~~ sign representation

$A_n \hookrightarrow S_n \twoheadrightarrow \mathbb{Z}/2 \subset \mathbb{C}^*$