Construction of V_n

$$\mathfrak{sl}_2\mathbb{C} \circlearrowright \mathbb{C}^2 = \mathrm{Span}\Big(\underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}}\Big).$$

$$\Rightarrow \circlearrowright \mathrm{Sym}^n\mathbb{C}^2 \simeq \mathbb{C}^{n+1},$$

h eigenvalues : sums of n eigenvalues in $\mathbb{C}^2$

$\Rightarrow$ $h \cdot v_1^{\otimes k} v_2^{\otimes n-k} = 2k-n$

range from $-n \quad -n+2 \quad \dots \quad +n$.

$$v_2^n \underset{e}{\overset{f}{\rightleftarrows}} v_2^{n-1}v_1 \underset{}{\overset{}{\rightleftarrows}} \cdots \underset{e}{\overset{f}{\rightleftarrows}} v_1^n$$

Let's consider instead $(\mathbb{C}^2)^* = \mathrm{Span}(x, y)$

$x\begin{pmatrix}a\\b\end{pmatrix} = a, \quad y\begin{pmatrix}a\\b\end{pmatrix} = b$ ie $x = (1\ 0), y = (0\ 1)$,

row vectors. --- linear functions on original $\mathbb{C}^2$

$$\mathrm{Sym}^n(\mathbb{C}^2)^* = \mathrm{Span}\{x^n \quad x^{n-1}y \quad \dots \quad y^n\}$$
$$= \text{homogeneous polynomials of deg } n \text{ on } \mathbb{C}^2.$$

Action sign: $\begin{pmatrix}a & b\\c & d\end{pmatrix} \cdot x = -(1\ 0)\begin{pmatrix}a & b\\c & d\end{pmatrix} = -(a\ b) = -(ax+by)$

$\rightsquigarrow$ $e \cdot y = 0 \quad e \cdot x = -y \qquad e = -y\frac{\partial}{\partial x} \in \mathrm{Vect}(\mathbb{C}^2)$

acts on $x^m y^n \longmapsto -m x^{m-1} y^{n+1}$.

Comes from differentiating $\exp(te) = \begin{pmatrix}1 & e^t\\0 & 1\end{pmatrix}$

$x \longmapsto x + e^t y$ pullback of function along group elt

$\frac{d}{dt}\big|_{t=0}$: $-ty = -y\frac{\partial}{\partial x}(x)$.

Similarly $f \cdot x = 0, \quad f \cdot y = -x, \quad f = -x\frac{\partial}{\partial y}$

$h \cdot x = -x \quad h \cdot y = +y \quad h = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$

(differentiate $\begin{pmatrix}e^t & \\ & e^{-t}\end{pmatrix}$)

So $\mathbb{C}[x,y]$ = Polynomial functions on $\mathbb{C}^2$
$= \bigoplus V_n$ direct sum of all irreps of $SL_2\mathbb{C}$
each appearing once! "model representation".

Note $SL_2\mathbb{C} \circlearrowright \mathbb{C}^2 \circlearrowleft \mathbb{C}^* = \begin{pmatrix} a & \\ & a \end{pmatrix}$ commuting actions
$SL_2\mathbb{C} \circlearrowright \mathbb{C}[x,y] \circlearrowleft \mathbb{C}^*$
Decompose into characters of $\mathbb{C}^*$
(holomorphic/algebraic characters $\mathbb{Z}^n \longleftrightarrow$
characters of $S^1 \subset \mathbb{C}^*$)
$\mathbb{C}[x,y] = \bigoplus \mathbb{C}[x,y]_k$, & each of these
is in turn $SL_2\mathbb{C}$ irreducible:
it's $SL_2\mathbb{C}$-preserved by commutation, but
irreducibility follows from dual pair property:

Prop $Aut_{SL_2\mathbb{C}}(\mathbb{C}[x,y]) = \mathbb{C}^*$ } each other's
$Aut_{\mathbb{C}^*}(\mathbb{C}[x,y]) = GL_2\mathbb{C}$ } centralizers.

Recall $\mathbb{C}^2 \setminus 0 = SL_2\mathbb{C}/N$
& note $\mathbb{C}[x,y] = \mathbb{C}[\mathbb{C}^2 \setminus 0]$ rational functions
regular on $\mathbb{C}^2 \setminus 0$ are same as polynomials on $\mathbb{C}^2$
by Hartogs theorem.

$\mathbb{C}^*$ action on $\mathbb{C}^2 \setminus 0$ doesn't come from (left) $SL_2\mathbb{C}$
action but rather $\text{Normalizer}_{SL_2\mathbb{C}}(N) = B = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \mathbb{C}^* \ltimes \mathbb{C}$
$\mathbb{C}P^1 = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^* = SL_2\mathbb{C}/B$
$G \circlearrowright G/K \circlearrowleft N_G(K)$ remnant of $G \circlearrowright G \circlearrowleft G$ commuting actions

$n \in \mathbb{Z} \rightsquigarrow$ line bundle $\mathcal{O}(n)$ on $\mathbb{CP}^1$
$\ell \longmapsto (\ell^*)^n = \mathrm{Sym}^n \ell^*$
Over $\mathbb{CP}^1 \setminus \infty$ & $\mathbb{CP}^1 \setminus 0$ identify $\mathcal{O}(n) \simeq$ trivial
thanks to sections x, y of $\mathcal{O}(1)$ $\dashrightarrow$ trivialize ℓ^*.

$\Rightarrow$ makes sense to discuss holomorphic/algebraic/C^∞...
sections of $\mathcal{O}(n)$: functions on
$\mathbb{CP}^1 \setminus \infty$ & $\mathbb{CP}^1 \setminus 0$ of desired type, s.t. on overlap
differ by $(z = \frac{x}{y})^n$.
$H^0(\mathbb{P}^1, \mathcal{O}(n)) = \Gamma(\mathcal{O}(n))$ = holomorphic sections of $\mathcal{O}(n)$:
on $\mathbb{C} = \mathbb{CP}^1 \setminus \infty$ get entire function f st f/z^n
regular at ∞ (entire on $\mathbb{CP}^1 \setminus 0$)
$\Rightarrow$ $f \in \mathrm{Span}\langle 1, z, z^2, \dots z^n\rangle = \mathrm{Span}\langle y^n, y^{n-1}x, \dots, x^n\rangle$
... all sections are algebraic, and come
from $\mathrm{Sym}^n V^* \twoheadrightarrow \ell^*$

$H^0(\mathbb{P}^1, \mathcal{O}(n)) = V_n = \mathrm{Sym}^n(\mathbb{C}^2)^*$ ($= 0$ $n < 0$)
... every irreducible representation appears.

In fact any $\mathcal{L} \to \mathbb{CP}^1$ holomorphic line bundle
is $\simeq \mathcal{O}(n)$ for some n ... tight fit with $SL_2\mathbb{C}$!
[determined by degree = # zeros − # poles of a section]
Borel-Weil-Bott Consider also $H^1(\mathbb{CP}^1, \mathcal{O}(n))$
= {sections on $\mathbb{CP}^1 \setminus \{0, \infty\}$} / sections on $\mathbb{CP}^1 \setminus 0$ ⊕ on $\mathbb{CP}^1 \setminus \infty$

z^{-n-1} z^{-n} z^{-n+1} .... z^{-1} 1 z z^2 z^3 .. z^{n-1} z^n z^{n+1}

$\mathcal{O}(k)$

$k = -n$ $\mathbb{C}[z^{-1}]z^k$ $H^1 \simeq V_{n-2}$

Serre duality: Canonical line bundle $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$
($dw = -\frac{1}{z^2}\,dz$ second order pole at ∞)

$$H^1(\mathbb{P}^1, \mathcal{L}) = H^0(\mathbb{P}^1, \mathcal{L}^* \otimes \omega)^*$$

$$\rightsquigarrow H^1(\mathbb{P}^1, \mathcal{O}(n)) = H^0(\mathbb{P}^1, \mathcal{O}(n-2))$$

$\cdots$	$\mathcal{O}(-4)$	$\mathcal{O}(-3)$	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\mathcal{O}(3)$	$\cdots$
	V_2	V_1	V_0	0	V_0	V_1	V_2	V_3	

$H^1 \neq 0 \quad H^0 = 0$ $\qquad\qquad$ $H^0 \neq 0 \quad H^1 = 0$

Why are these spaces representations to begin with?

Induced Representations

G finite $\supset H$ subgroup. Construct G-reps from H-reps:

$$\mathrm{Ind}_H^G : \mathrm{Rep}\, H \longrightarrow \mathrm{Rep}\, G \quad \text{induction functor}$$
$$W \longmapsto V = \mathrm{Ind}_H^G W$$

As a vector space, $V \simeq \bigoplus_{[g] \in G/H} [g] \cdot W$ copy for every coset.

Def: $\mathrm{Ind}_H^G W = \mathrm{Map}_H(G, W)$
$= \{ f : G \to W$ W-valued function on G
s.t. $f(gh^{-1}) = h \circ f(g) \}$

$g \in G$ acts by left translation on G, $g \cdot f(g_1) = f(gg_1)$.
Value of f on each H-coset is determined by value on one representative $\rightsquigarrow \bigoplus [g] \cdot W$.

Group algebra interpretation $\mathrm{Map}(G, W) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}G, W)$
so induced rep is a subspace of this Hom:

$\mathbb{C}G$ is an H-module via $H \hookrightarrow G$ right action

$\mathrm{Ind}_H^G W = \mathrm{Hom}_H(\mathbb{C}G, W)$ H-rep = $\mathbb{C}H$-module maps

Calculate $\mathrm{Hom}_G(V, \mathrm{Hom}_H(\mathbb{C}G, W))$
for any $V \in \mathrm{Rep}\, G$

$= $ Subspace of $\mathrm{Hom}(V, \mathrm{Hom}(\mathbb{C}G, W))$ with $G \times H$ invariance

(G, H actions commute: H acting from right on $\mathbb{C}G$)

$= \mathrm{Hom}(\mathbb{C}G \otimes V, W)^{G\times H}$

$= \mathrm{Hom}_G(\mathbb{C}G, \mathrm{Hom}_H(V, W))$.

But $\mathrm{Hom}_G(\mathbb{C}G, \text{any } U) = U$ (free rank 1 module)

$\Rightarrow$ ~~Frobenius Reciprocity~~

$$\boxed{\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) = \mathrm{Hom}_H(\mathrm{Res}_G^H V, W)}$$

left adjoint to restriction.

Another point of view

R ring (not nec. commutative), M_R, ${}_R N$ right & left modules

$\rightsquigarrow M \otimes_R N = M \otimes N / mr \otimes n = m \otimes rn$

ie source of R-bilinear maps out of $M \times N$.

$$\begin{tikzcd} M \times N \arrow[rr] \arrow[dr] & & P \quad R\text{-bilinear} \\ & M \otimes_R N \arrow[ur] & \end{tikzcd}$$

Claim $\mathrm{Ind}_H^G W \cong \underbrace{\mathbb{C}G \otimes_{\mathbb{C}H} W}_{\text{coinduction}} = \bigoplus_{g\in G} g \cdot W \Big/ gh\, w = g(hw)$

[Why these two opposite pictures? $\delta_g \in \mathbb{C}G$ plays both role of function (d-f to $g \in G$) & atomic measure on G .....
$\mathbb{C}G$ has nondegenerate inner product s.t. $\langle \delta_g, f \rangle = f(g)$:
$\langle f, h \rangle = f * h(1)$ value at $1 \in G$.]

So $\mathrm{Hom}_G(\mathrm{Ind}_H^G W, V) = \mathrm{Hom}_G(\mathbb{C}G \otimes_{\mathbb{C}H} W, V)$
$\mathrm{Hom}_G(\mathbb{C}G, \mathrm{Hom}_H(W, V)) = \mathrm{Hom}_H(W, V)$

$\Rightarrow$ $\boxed{\mathrm{Hom}_G(\mathrm{Ind}_H^G W, V) = \mathrm{Hom}_H(W, \mathrm{Res}_G^H V)}$
second Frobenius reciprocity! (2-sided adjoints)
(reciprocity for "coinduction")

<u>Geometric interpretation</u> $G \supset H \circlearrowright W$
$\rightsquigarrow$ construct vector bundle w/ fibers $\simeq W$ over G/H:
G H-bundle. $\rightsquigarrow$ take "associated vector bundle
$\downarrow^H$
G/H $\quad G \times_H W = G \times W / (g, w) \sim (gh^{-1}, hw)$
$\qquad (gh, w) \sim (g, hw)$

Fibers over any coset $\simeq W$, a choice of coset rep gives such an isomorphism $\rightsquigarrow$ vector bundle.

$G \times_H W$ is a G-equivariant vector bundle on G/H:

$G \circlearrowright \overset{V}{\downarrow} X$ is G-equivariant if we're given $\varphi_g : g^* V \xrightarrow{\sim} V$
$\forall g \in G$ (ie. $\varphi_g : V|_{gx} \xrightarrow{\sim} V|_x$) & $\varphi_{gh} = \varphi_h \varphi_g$ associativity.

$$\begin{array}{c} V \\ \downarrow \\ G \curvearrowright X \end{array}$$ G-equivariant $\rightsquigarrow$ sections of V, $\Gamma(V)$ (assignments $x \mapsto f(x) \in V|_x$) form a representation of G:

$g \cdot f(x) = \varphi_g(f(gx))$.

$X = G/H$, $V = G \times_H W \rightsquigarrow V$ is G-equivariant

$k \cdot (g \times w) = kg \times w \sim kgh^{-1} \times hw$

preserves equivalence relation

& $\Gamma(V) = \mathrm{Map}_H(G, W) = \mathrm{Ind}_H^G W$.

$G \supset H$ Lie groups: G/H is a homogeneous space (manifold w/ ~~transitive~~ G-action).

$H \curvearrowright W$ f.d. rep $\rightsquigarrow$ $G \times_H W$ is a ~~smooth~~ vector bundle over G/H ... why locally trivial?

Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ as vector spaces

$\mathfrak{g}$ = Lie G, $\mathfrak{h}$ = Lie H $\to$ near $1 \in G$ can uniquely write any

$g = \exp(s)\exp(h)$ $s \in \mathfrak{s}$, $h \in \mathfrak{h}$

$\rightsquigarrow$ open neighborhood of $[H] \in G/H$ isomorphic to open neighborhood of $0 \in \mathfrak{s}$

$\rightsquigarrow$ section of $\begin{array}{c} G \\ \downarrow \\ G/H \end{array}$ near basepoint $\rightsquigarrow$ write $G \sim G/H \times H$ locally

$\Rightarrow$ trivialize bundle....

$\Rightarrow$ lots of analogs of Induction: smooth sections, cptly supp smooth sections, L^p, analytic, algebraic, ... whatever structures G/H carries!

Our case: $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \simeq \mathbb{C}^* \ltimes \mathbb{C} \subset SL_2\mathbb{C}$

Let's induce 1-dimensional reps of B

[NB any <u>algebraic</u> irrep of B is 1-dimensional: $\mathbb{C} \simeq N$ must act by e^{itx}, to be polynomial need $t=0$... $\mathbb{C}^* \subset B$ preserves N-invariants $\rightsquigarrow$ whole rep is N-invariants $\rightsquigarrow$ 1-dimensional]

$B \longrightarrow \mathbb{C}^* = GL_1(\mathbb{C}) \circlearrowright \mathbb{C}$ 1-dim rep

$\searrow B/[B,B] = H$ given by $\chi : H \longrightarrow \mathbb{C}^*$

$\quad\quad \overset{\shortparallel}{N}$ $\quad\quad z \longmapsto z^n$ some $n \in \mathbb{Z}$.

$\rightsquigarrow \mathbb{C}_\chi$ irreps of B.

$\mathrm{Ind}_B^G \mathbb{C}_\chi$: sections of line bundle $SL_2\mathbb{C} \times_B \mathbb{C}_n$

$= \underset{\mathbb{C}^2 - 0}{\overset{\shortparallel}{SL_2\mathbb{C}/N}} \times_H \underset{\mathbb{C}^*}{\overset{\shortparallel}{}} \mathbb{C}_n = \mathcal{O}(-n)$ ← $\mathbb{C}^*$ acts on l with weight 1...

holomorphic sections: "holomorphic induction".

– become holomorphic functions [in fact polynomials] on $SL_2\mathbb{C}/N$ since bundle is trivial here.

$\rightsquigarrow \{ f \in \mathbb{C}[SL_2\mathbb{C}/N] : f(xh) = h^n f(x) \}$

[Harish-Singer: use smooth inductions $C^\infty(S^2)$ to good effect...]

__Reverse engineering Borel-Weil__

V irrep of $SL_2\mathbb{C}$, write $V^* = \Gamma(\mathbb{P}(V), \mathcal{O}(1))$

sections of dual to tautological line bundle give linear coords on V

Suppose $v \in V$ a highest weight vector:

$x \cdot v = 0 \quad hv = \lambda v \implies N \cdot v = v \quad B \cdot \mathbb{C}v = \mathbb{C}v$

$\rightsquigarrow [v] \in \mathbb{P}(V)$ is a B-fixed point.

orbit $\mathcal{O}_v = SL_2\mathbb{C} \cdot [v] \simeq SL_2\mathbb{C}/B = \mathbb{P}^1$: $\mathrm{Lie}(\mathrm{Stab}[v]) = \mathfrak{b}$

& $\mathbb{C}\mathbb{P}^1$ has no covering spaces...

So $\mathcal{O}_v \simeq \mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^n$,

V irreducible $\implies$ $\mathcal{O}_v$ not contained in any hyperplane in $\mathbb{C}\mathbb{P}^n$ (span V)

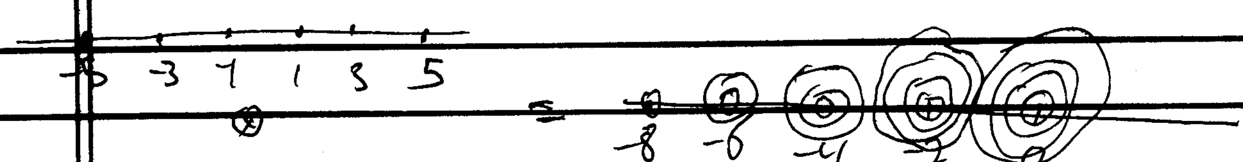
$\rightsquigarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^n}(1)|_{\mathbb{P}^1}) \simeq V^*$

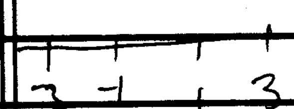
... find $\mathcal{O}_v$ has degree n in $\mathbb{C}\mathbb{P}^n$: rational normal curve

... image of Veronese embedding $\mathbb{P}(W) \hookrightarrow \mathbb{P}(\mathrm{Sym}^n W)$

$[w] \mapsto [w^n]$, or in bases $[x, y] \mapsto [x^n, x^{n-1}y, x^{n-2}y^2 \ldots y^n]$

__Decomposing reps__ What is $\mathrm{Sym}^5 V \otimes \mathrm{Sym}^3 V$ as $SL_2\mathbb{C}$ rep





$V_8 \oplus V_6 \oplus V_4 \oplus V_2$

$V_n \otimes V_m \twoheadrightarrow V_{n+m}$ ring structure on $\bigoplus_n V_n = \mathbb{C}[x, y]$

Universal Enveloping Algebras

$G : \mathbb{C}G :: \mathfrak{g} : U\mathfrak{g}$:

construct associative algebra with same representation theory

$\mathrm{Hom}_{Lie}(\mathfrak{g}, \mathfrak{gl}(V)) = \mathrm{Hom}_{Alg}(U\mathfrak{g}, \mathrm{End}(V))$.

Recall A associative algebra $\rightsquigarrow A^{Lie} = A, [a,b] = ab - ba$

Lie algebra $\rightsquigarrow$ functor $\mathrm{Lie} : \mathrm{Assoc} \longrightarrow \mathrm{Lie}$

Can ask for a functor $U : \mathrm{Lie} \longrightarrow \mathrm{Assoc}$ s.t.

$\mathrm{Hom}_{Lie}(\mathfrak{g}, A^{Lie}) = \mathrm{Hom}_{Assoc}(U\mathfrak{g}, A)$

(left adjoint to Lie) — universal property, uniquely characterizes U if exists.

Construction of $U\mathfrak{g}$

V vector space $\rightsquigarrow TV$ tensor algebra on V

$TV = \bigoplus V^{\otimes n}$, $\otimes$ free associative algebra on V

$\mathrm{Hom}_{Vect}(V, \mathrm{Forget}(A)) = \mathrm{Hom}_{Assoc}(TV, A)$

Now for $V = \mathfrak{g}$ Lie algebra, just need to enforce $\mathfrak{g}$ relations to make maps out of TV into Lie algebra maps.

$U\mathfrak{g} = T\mathfrak{g} \,/\,$ 2-sided ideal generated by $\underbrace{xy - yx}_{\in\, \mathfrak{g}^{\otimes 2}} = \underbrace{[x,y]}_{\in\, \mathfrak{g}^{\otimes 1}}$, $x, y \in \mathfrak{g}$

eg $[\,,\,] = 0$ ($\mathfrak{g}$ abelian) $\Rightarrow U\mathfrak{g} = \mathrm{Sym}\,\mathfrak{g}$

The relation is automatically satisfied by operators in any rep of $\mathfrak{g}$ $\rightsquigarrow$ extend to reps of $U\mathfrak{g}$.

Q: How big is $U\mathfrak{g}$? A: Like $\mathrm{Sym}\,\mathfrak{g}$.

PBW theorem: $U\mathfrak{g} \xrightarrow{\sim} \mathrm{Sym}\,\mathfrak{g}$ as vector spaces given ordered basis of $\mathfrak{g} = \mathrm{Span}\langle x, y, z, \ldots \rangle$

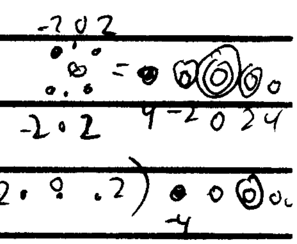
– prove by induction on degree of an expression

$\ldots yx \ldots = \ldots xy \ldots - \ldots [x,y] \ldots = \ldots xy \ldots - \sum \ldots c_{xy}^{w} w \ldots$

which by induction can be written in lexicographic order $\quad ([x,y] = \sum c_{xy}^{w} w)$

In fact $U\mathfrak{g}$ has a natural algebra filtration $U\mathfrak{g}_{\leq n}$, $U\mathfrak{g}_{\leq m} \longrightarrow U\mathfrak{g}_{\leq n+m}$, & associated graded algebra $\mathrm{gr}\, U\mathfrak{g} = \bigoplus U\mathfrak{g}_{\leq n} / U\mathfrak{g}_{\leq n-1} \xrightarrow{\sim} \mathrm{Sym}\,\mathfrak{g}$ as algebras.

$$\begin{array}{lllllll} & e^2 & eh & \overset{h^2}{ef} & hf & f^2 & \mathrm{Sym}^2\mathfrak{g} = V_4 \oplus V_0 \\ U\mathfrak{sl}_2: & e & h & f & & & \mathfrak{g} = V_2 \\ & & 1 & & & & \end{array}$$



$\ldots$ expressions of degree n in lexicographic basis [illegible]

(ie $\mathrm{Gr}_n\, U\mathfrak{sl}_2$) $\simeq \mathrm{Sym}^n \mathfrak{g}$ & is a $\mathfrak{g}$-rep!

~~deg(xw − wx) = deg wx~~ (but might not sit in fixed degree n in our basis, only in assoc. graded).

Crucial observation: there is a $\mathfrak{g}$-invariant element (! up to scalars) in $U\mathfrak{g}_2$ --- a linear combo of ef & h^2 in $\mathrm{Sym}^2\mathfrak{g}$

~ the Casimir element $C = ef + fe + \frac{1}{2}h^2$

$(= 2ef + \frac{1}{2}h^2 - h$ in lexico basis$)$

Where does Casimir come from?

$(\mathrm{Sym}^2 \mathfrak{g})^{\mathfrak{g}}$ = invariant bilinear forms on $\mathfrak{g}$

$\langle [x,u],v\rangle + \langle u,[x,v]\rangle = 0$

(derivative of $\langle gu, gv\rangle = \langle u,v\rangle$)

$\mathfrak{g}$ irreducible rep of itself $\rightsquigarrow$ invariant nondeg form is unique/scalar:
[simple Lie algebra]

$\langle , \rangle$ induces $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ as $\mathfrak{g}$-reps

unique up to scalar by Schur's lemma.

$\mathfrak{sl}_2$: $\langle x,y\rangle = \mathrm{Tr}(xy)$ is this form (/scalar)

Now want to lift to invariant in $U\mathfrak{g}_{\leq 2}$:

let $C = \sum e_i e^i \in U\mathfrak{g}_{\leq 2}$ for any basis e_i
& dual basis e^i

$\langle e,f\rangle = \mathrm{Tr}\begin{pmatrix} & 1\\ & \end{pmatrix}\begin{pmatrix} & \\ 1 & \end{pmatrix} = 1$, $\langle h,h\rangle = \mathrm{Tr}\begin{pmatrix} 1 & \\ & -1\end{pmatrix}^2 = 2$

$\rightsquigarrow C = ef + fe + \frac{1}{2}h^2$

[can check $Ce = efe + fee + \frac{1}{2}h^2 e$
$= (eef - eh) + (efe - he) + \frac{1}{2}(heh - he)$
$= eef + efe - eh + (\frac{1}{2}eh^2 + eh)$
$= eef + efe + \frac{1}{2}eh^2 = eC$ ...]

What does $U\mathfrak{g}^{\mathfrak{g}}$ mean? commuting with $\mathfrak{g}$

$\longleftrightarrow$ commuting with all $U\mathfrak{g}$

$\Rightarrow$ $U\mathfrak{g}^{\mathfrak{g}} = Z(U\mathfrak{g})$ center of the assoc. alg. $U\mathfrak{g}$

Schur's lemma Any $z \in Z$ (eg Casimir) acts as a scalar in any irrep (~~[illegible]~~) of $\mathfrak{g}$.

eg Casimir on $V^{(n)}$:

$e\cdot v_n = 0 \quad h\cdot v_n = n v_n$

$C\cdot v_n = ef v_n + \cancel{fe v_n} + \frac{1}{2} h^2 v_n$

$= (fe v_n + h v_n) + \frac{1}{2} n^2 v_n$

$= n v_n + \frac{1}{2} n^2 v_n = \boxed{n\left(\frac{n}{2}+1\right)} = \frac{1}{2}(n+1)^2 - \frac{1}{2}$

$\rightsquigarrow$ can decompose any f.d. sl_2 irrep just using Casimir: $V = \bigoplus_n \underbrace{V_n^{\oplus \mu_n}}_{C \text{ eigenspace w/ eigenvalue } n(\frac{n}{2}+1)}$

<u>Proposition</u> $Z sl_2 \simeq \mathbb{C}[C]$: center is given just by powers of C.

Sketch: need to calculate $(\mathrm{Sym}^n \mathfrak{g})^{\mathfrak{g}} \simeq \begin{cases} \mathbb{C} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

... count 0 & 2 weight spaces in $\mathrm{Sym}^n \mathfrak{g}$,

see if same number or one more 0.

-2 0 2
-2 0 2
-2 0 2
-2 0 2
-2 0 2

<u>Harish-Chandra isomorphism</u>

$Z sl_2 \simeq \mathbb{C}[\underbrace{\mathfrak{h}^*}_{\text{possible eigenvalues of } h}]^{\mathbb{Z}/2}$ where $W = \mathbb{Z}/2$ acts as $\lambda \mapsto -\lambda - 2$ (reflection in $\lambda = -1$)

($\mathfrak{h}^*/\mathbb{Z}/2 \simeq \mathbb{C}$ via coordinate $(\lambda+1)^2$ ...

Note we already saw $\lambda \mapsto -\lambda - 2$ symmetry: as Serre duality on $\mathbb{CP}^1$, $\mathcal{O}(n) \mapsto \mathcal{O}(-n-2)$ ...