The Casimir & Harmonic Analysis

$\mathfrak{g}$ = left invariant vector fields on G

$U\mathfrak{g}$ - left invariant diffops on G:

generated by products of left invariant vector fields

— eg if $G \simeq \mathbb{R}^n/\Lambda$ abelian

these are just constant coefficient diffops.

Left invariant vector fields come from differentiating right action of G : $\mathrm{Aut}_{G_L}(G) = G_R$ ...

A differential operator is bi-invariant $\longleftrightarrow$ central in $U\mathfrak{g}$

$\boxed{Z\mathfrak{g} = \mathcal{D}(G)^{G\times G}}$ bi invariant diffops

Casimir C: canonical bi-invariant diffop of degree two on G with nondeg. invariant form (eg $Sl_2\mathbb{C}$) : a form of the Laplace-Beltrami operator

$Z\mathfrak{g}$ (or C) bi-invariant $\rightsquigarrow$ descends to give diffops on any homogeneous space G/H, still left invariant

Can decompose functions on G/H into eigenspaces of $Z\mathfrak{g}$ (i.e of C for $Sl_2\mathbb{C}$) $\rightsquigarrow$ in our case this gives the decomposition into irreps (isotypic components)

... ie performs 'harmonic analysis' on G/H ...

Let's see this in action...

[**Remark** Any f.d. rep of $Sl_2\mathbb{C}$ depending continuously on parameters must be constant/iso: eigenvalues of Casimir continuous → constant!!]

Most basic case: functions on G itself.

Let $\mathbb{C}[G]$ denote polynomial (regular) functions on G.

$SL_2\mathbb{C} = \{ad - bc = 1\} \subset Mat_{2\times 2} = \text{End}\,\underset{\overset{\circlearrowright}{SL_2\mathbb{C}}}{\mathbb{C}^2} = \mathbb{C}^2 \otimes \underset{\overset{\circlearrowleft}{SL_2\mathbb{C}}}{\mathbb{C}^{2*}}$

$\rightsquigarrow \mathbb{C}[SL_2\mathbb{C}] = \mathbb{C}[a,b,c,d] / ad-bc=1$

$= \bigoplus \text{Sym}^n(\mathbb{C}^2 \otimes \mathbb{C}^{2*}) \Big/ \langle ad-bc=1 \rangle$

[$G = \overset{\text{reductive}}{\text{complexification}}$ of compact K (G simply connected, $\mathfrak{g} = \text{Lie}\, \mathfrak{g} = \text{Lie}\, K \otimes \mathbb{C}$) $\rightsquigarrow$

$\mathbb{C}[G] = L^2(K)^{fin}$ finite $K\times K$ vectors]

Peter-Weyl Theorem $\mathbb{C}[SL_2\mathbb{C}] \simeq \bigoplus_{n\in\mathbb{Z}_+} V_n \otimes V_n^*$

as $SL_2\mathbb{C} \times SL_2\mathbb{C}$ reps

$\mathbb{C}[G] = \bigoplus_{V \text{ irrep}} V \otimes V^*$

Construction via matrix elements:

$V \otimes V^* \xrightarrow[G\times G \text{ map}]{} \mathbb{C}[G]$ $\quad v \otimes v^* \mapsto \underbrace{\{g \mapsto \langle v^*, g\cdot v\rangle\}}_{f_{v,v^*}}$

V irrep $\Rightarrow V \otimes V^*$ $G\times G$ irrep $\Rightarrow$ this is an inclusion.

[check $(h_1 \times h_2) f_{v,v^*}(g) = \langle v^*, h_1^{-1} g h_2 \cdot v\rangle =$

$\langle h_1 v^*, g\cdot(h_2 v)\rangle = f_{h_2 v, h_1 v^*}(g)$

$\text{End}_{G\times G}(V\otimes V^*) = (\text{End}\, V)^G \otimes (\text{End}\, V^*)^G = \mathbb{C}$]

Claim this is entire V-isotypic component of $\mathbb{C}[G]$,

ie $V \otimes V^* \simeq V \otimes \text{Hom}(V, \mathbb{C}[G]) \hookrightarrow \mathbb{C}[G]$

Cheap argument for $SL_2\mathbb{C}$: V_n isotypic component is precisely $C(V_n)$ eigenspace for Casimir ~ which is same for left & right G actions. In particular it's preserved by symmetry $g \mapsto g^{-1}$ of G, switching G_ℓ & G_r, So (up to $V \mapsto V^*$) multiplicity of V & V are exchanged.

[Alternatively: have nondegenerate inner product carry from trace $\mathbb{C}[G] \longrightarrow \mathbb{C}$, $f \mapsto f(1)$

$f_{v,v^*} \longmapsto \langle v^*, v\rangle$ nondegenerate on $V \otimes V^*$

G finite: use nondegeneracy of $\langle f, g\rangle \mapsto tr(f * g)$ on group algebra]

Algebra of matrix elements:

$$f_{V,v,v^*} + f_{W,w,w^*} = f_{V\oplus W, v\oplus w, v^*\oplus w^*}$$

$$f_{V,v,v^*} \cdot f_{W,w,w^*} = f_{V\otimes W, v\otimes w, v^*\otimes w^*}$$

$\rightsquigarrow$ ring structure on $R = \bigoplus V \otimes V^*$.

Alternatively: show matrix elts span: $W = \mathrm{Span}(f_1 \ldots f_n) \subset \mathbb{C}[G]$ G-invariant $\Rightarrow g \cdot f_i = \sum M_{ij}(g) f_j$. But $f_i(g) = (g^{-1} \cdot f_i)(1) = \sum M_{ij}(g^{-1}) f_j(1)$ $\Longrightarrow f_i$ is linear combo of matrix elts $M_{ij}^*(g) = M_{ij}(g^{-1})$ for dual rep W^*

Case of $SL_2\mathbb{C}$: can see Peter-Weyl by:

1. Check $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^{2*}] \twoheadrightarrow R$ since $\mathbb{C}^2$ generates all reps of $SL_2\mathbb{C}$
2. Check $ad-bc \longmapsto 1$ from fact that $SL_2\mathbb{C}$ acts on $\mathbb{C}^2$ with determinant one ($\Lambda^2\mathbb{C}^2$ trivial)

$\rightsquigarrow$ get $\mathbb{C}[a,b,c,d]/ad-bc=1 \twoheadrightarrow R$.

Now check dimensions of (natural) graded pieces agree $\rightsquigarrow$ injective.

Consequences of Peter-Weyl

$$\mathbb{C}[G/H] = \mathbb{C}[G]^H = \bigoplus_V V \otimes (V^*)^H$$

We know any irrep has 1-dimensional N invariants (highest weight)
$\rightsquigarrow$ recover $\mathbb{C}[G/N] = \bigoplus_{V \text{ irrep}} V = \bigoplus_{n} V_n$.

Similarly $\mathbb{C}[G/\underset{\mathbb{C}^*}{H}] = \bigoplus_n V_n \otimes (V_n)_0$ eigenspace

$$= \bigoplus_{\ell \in \mathbb{Z}} V_{2\ell} \quad \text{all even reps}$$

real alg = functions on real varieties & $S^2, S^3 \otimes \mathbb{C}$

Consider $S^2 = SU_2/T = SO_3/SO_2$
let $\mathbb{C}[\;]$ denote restriction of polynomial functions on complexification ... $\mathbb{C}[SU_2] = \mathbb{C}[SL_2\mathbb{C}] = \bigoplus V_n \otimes V_n^*$

$$\mathbb{C}[S^2] = \mathbb{C}[SU_2]^T = \mathbb{C}[SL_2\mathbb{C}]^{\mathbb{C}^*} = \bigoplus_\ell V_{2\ell}.$$

Casimir interpretation: $\mathbb{C}[SU_2]$ means $\bigoplus$ of eigenspaces of C.
What is C on SU_2: $\mathbf{i} = \begin{pmatrix} i & \\ & -i \end{pmatrix}$ $\mathbf{j} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\mathbf{k} = \begin{pmatrix} & i \\ -i & \end{pmatrix}$
$\rightsquigarrow h = -i\,\mathbf{i}$, $e = \frac{1}{2}(\mathbf{j} - i\mathbf{k})$, $f = -\frac{1}{2}(\mathbf{j} + i\mathbf{k})$

$$C = ef + fe + \tfrac{1}{2}h^2 = -\tfrac{1}{2}(\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2) \quad \text{obvious quadratic}$$

On S^2: $C = \cancel{\tfrac{1}{2}}\Delta$ spherical Laplacian (up to a scalar) ~ factor of 2

$C^\infty(S^2) \supset H = \bigoplus(\Delta \text{ eigenspaces})$ "algebraic fns"

$$H = \bigoplus_{\ell} H_\ell = \{ f : \Delta f = \ell(\ell+1) f \}$$

$\frac{1}{4}((n+1)^2-1)$ $n = 2\ell$

Each H_ℓ is a sum of $V_{2\ell}$'s just by eigenvalue
– but in fact by Peter-Weyl $H_\ell \cong V_{2\ell}$.

Further decomposition $H_\ell = \bigoplus_{m=-\ell}^{\ell} H_\ell^m = \{ \mathbf{i} f = imf \}$
eigenspaces of angular momentum around x axis:

$$\mathbf{i} \longleftrightarrow (\cdot, -1) \longleftrightarrow y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \frac{\partial}{\partial \varphi} \text{ (spherical coords)}$$

$H_\ell^m = \mathbb{C} \cdot Y_\ell^m$ spherical harmonics, form basis for functions on S^2. (dense in C^∞ or L^2..)

multiplicity one can also be seen analytically:
Check $H_\ell^0 \leq 1$-dimensional: function invariant under x rotation & satisfying $(\Delta - \ell(\ell+1))f = 0$ determined by restriction to x=const hyperplane
((Cauchy-Kovalevsky theorem))

Separation of variables for Laplacian on $\mathbb{R}^3$: $f \in C^\infty(S^2)$
eigenfunction of Δ_{S^2} extends uniquely to (a ball in)
$\mathbb{R}^3 \setminus 0 = S^2 \times \mathbb{R}_+$ as harmonic function $\Delta_{\mathbb{R}^3} \tilde{f} = 0$
..... Laplace eqn becomes ODE for radial dependence of f given $(\Delta_{S^2} - \lambda) f = 0$

In fact $\mathbb{C}[x,y,z] \simeq \underset{\text{radial polynomials}}{\mathbb{C}[r^2]} \otimes H$ harmonic polynomials

$\nwarrow \{ f \in \mathbb{C}[x,y,z] : \Delta f = 0 \}$

$r^2 = x^2 + y^2 + z^2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$= \mathbb{C}[\mathbb{R}^3]$

Algebraic version $G \circlearrowright \mathbb{C}^3$ (complex version of our $\mathbb{R}^3$.)

Look at $\mathbb{C}[\mathfrak{g}^*]$ as G-representation

$= \mathrm{Sym}\,\mathfrak{g}$ ... "classical" version of $U\mathfrak{g}$.

[identify $\mathfrak{g} \simeq \mathfrak{g}^*$]

$\mathbb{C}[\mathfrak{g}]^G$ (analog of center of $U\mathfrak{g}$)

$\simeq \mathbb{C}[\mathfrak{h}]^W$ (Chevalley restriction theorem)

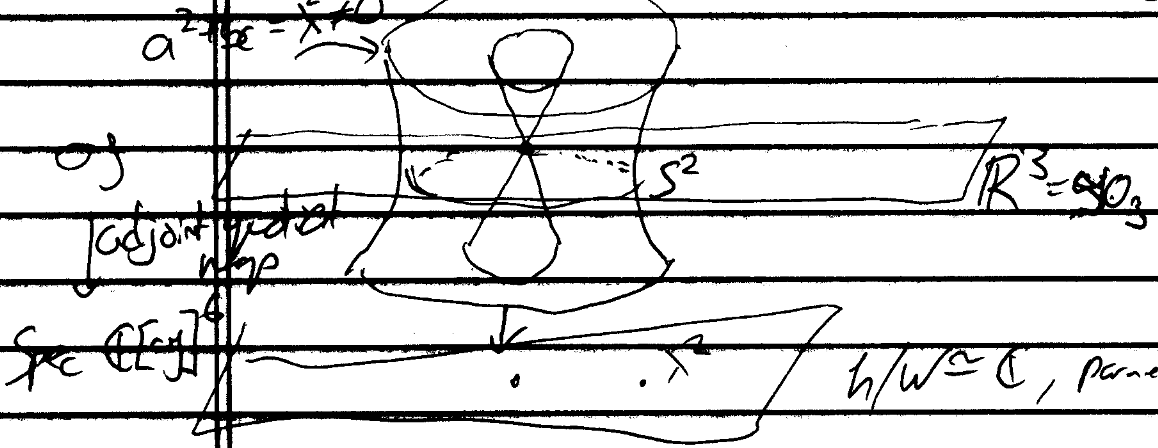
... follows from Jordan decomposition:

$\begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$ conjugate to $\begin{pmatrix}\lambda & \varepsilon\\ & \lambda\end{pmatrix}$ $\forall \varepsilon \neq 0$

So any continuous G-invariant function determined by values on diagonal matrices, which must be $W = \mathbb{Z}/2$ symmetric. Conversely any $\in \mathbb{C}[\mathfrak{h}]^W$ extends to G: as coefficients of characteristic polynomial.

$\mathbb{C}[\mathfrak{sl}_2]^{SL_2} = \mathbb{C}[a^2+bc]$: $-\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \frac{1}{2}\operatorname{tr}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^2$

$= a^2+bc$ when $\operatorname{tr}=0$.



$\mathfrak{h}/W \simeq \mathbb{C}$, parameter $a^2+bc = \lambda^2$, $\lambda \in \mathfrak{h}^*$

Generic fiber = semisimple conjugacy class: $SL_2\mathbb{C}\cdot\begin{pmatrix}\lambda & \\ & -\lambda\end{pmatrix} = \mathbb{O}_{\lambda^2}$

$\simeq SL_2\mathbb{C}/\mathbb{C}^*$. Complexification of

$S^2 = SU_2/T$, real slice.

Peter-Weyl: $\mathbb{C}[\mathbb{O}_{\lambda^2}] \simeq \bigoplus_{\ell\in\mathbb{Z}} V_{2\ell}$

Most interesting fiber (not seen in SU_2 slice):

<u>nilpotent cone</u> $\mathcal{N} = \pi^{-1}(0) = \{a^2+bc=0\}$

$$\begin{tikzcd}[column sep=small] = SL_2\mathbb{C}\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} \arrow[d] \amalg \{0\} & & \\ SL_2\mathbb{C}/\begin{pmatrix}\pm1&*\\0&\pm1\end{pmatrix} = PSL_2\mathbb{C}/N \arrow[r, two heads] & \mathbb{P}^1 = SL_2\mathbb{C}/B \end{tikzcd}$$

fiber over a line ℓ is $(\ell/\pm)\setminus 0 \cong \ell^2\setminus 0$

∴ orbit is total space of $\mathcal{O}(-2) = T^*\mathbb{P}^1$ minus zero section.

$\rightsquigarrow \mathcal{N} \simeq T^*\mathbb{P}^1$ with zero section contracted to a point.

In fact have canonical resolution of singularities

Springer resolution

$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d] & = \{x\in\mathcal{N} + \text{line } \ell\in\mathbb{P}^1 \text{ preserved by } x\} \arrow[d] \\ \mathcal{N} & \mathbb{P}^1 \end{tikzcd}$$

$: \tilde{\mathcal{N}} \simeq T^*\mathbb{P}^1$ ∴

$\mathbb{P}^1 = SL_2\mathbb{C}/B, \quad T_\infty\mathbb{P}^1 = \mathfrak{sl}_2\mathbb{C}/\mathfrak{b} \quad (\bullet)$

$\rightsquigarrow T^*_\infty\mathbb{P}^1 \simeq \mathfrak{n} \quad (*)$.

$=$ nilpotents preserving line ∞

Hartogs extension: $\mathbb{C}[\mathcal{N}] \xrightarrow{\sim} \mathbb{C}[SL_2\mathbb{C}\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}]$ open orbit

$= \mathbb{C}[(\mathbb{C}^2\setminus 0)/\pm1] =$ even degree homogeneous polynomials $= \bigoplus_{\ell\geq 0} V_{2\ell}$

So we've seen explicitly functions on any fiber $\simeq H$

Algebraic separation of variables:

Theorem (Kostant) $\mathbb{C}[\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g}]^G \otimes H.$
Wallach
$\leftarrow$ fibers

How to pick out functions on fibers? harmonic polynomials

Setup: G group $\curvearrowright V$ vector space

Let $\mathcal{D}_V$ = constant coefficient (= translation inv't) differential ops on $V \simeq \mathrm{Sym}\, V$, commutative algebra

Dual to $\mathbb{C}[V] = \mathrm{Sym}\, V^*$ via $D \otimes P \longmapsto DP(0)$

... if have same degree $\Rightarrow$ $DP = DP(0)$ is a constant, other components orthogonal.

$P \in \mathbb{C}[V]$ is G-harmonic if $DP=0$ $\forall D \in \mathcal{D}_V^G$ with vanishing constant term. $\mathcal{H}$ = all harmonic polynomials

G = trivial $\rightsquigarrow$ constants

$G = SO_n$ get $\mathcal{D}_V^G = \mathbb{C}[\Delta]$, so G-harmonic means harmonic.

Theorem G reductive (f.d. reps are semisimple — eg finite or $SL_2\mathbb{C}$)

& $\mathbb{C}[V]$ free as $\mathbb{C}[V]^G$-module

$\Rightarrow$ $\mathbb{C}[V] \simeq \mathbb{C}[V]^G \otimes \mathcal{H}.$

Our case freeness automatic: $\mathbb{C}[V]^G \simeq \mathbb{C}[X]$, polynomials in one variable, so since our module has no torsion $\rightsquigarrow$ free

$\mathcal{D}(\mathfrak{sl}_2)^{SL_2} \simeq \mathrm{Sym}\, \mathfrak{g}^G$ generated by a^2+bc ... Laplacian!

[intuition: G-harmonic $\longleftrightarrow$ constant along "radial" direction - i.e. over V/G (in $\mathbb{C}[V]^G$ coords).

so expect to look like functions on a fiber.]

Characters

V rep $\qquad V \otimes V^* \simeq \operatorname{End} V \longrightarrow \mathbb{C}[G]$

$$\overset{\in}{\mathrm{Id}_V} \longmapsto \overset{\in}{\chi_V}$$

character of V.

In bases e_i, e^i of V, V^* : $\mathrm{Id}_V = \sum e_i \otimes e^i$

$\rightsquigarrow \chi_V(g) = \sum_i \langle e^i, g \cdot e_i \rangle = \mathrm{tr}_V(g)$

trace of g in representation V

eg $\chi_V(1) = \dim V$.

{Note for V irreducible $\mathbb{C}\,\mathrm{Id}_V \subset V \otimes V^*$ is precisely subspace of diagonal G invariants : $(V \otimes V^*)^{G_{\mathrm{diag}}} = \operatorname{End}_G V = \mathbb{C}\,\mathrm{Id}_V$}

$\rightsquigarrow \mathbb{C}[G]^{G_{\mathrm{diag}}}$ class functions

$= \bigoplus_{V \text{ irrep}} \mathbb{C} \chi_V$ sum of characters.

So characters of irreps linearly independent & span class functions

($\rightsquigarrow$ irreps are <u>determined</u> by their characters :

$\chi : \{\text{irreps}\} \hookrightarrow \{\text{class functions}\}$)

Consider χ_V on $H = \left\{\begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}\right\} \simeq \mathbb{C}^* \subset SL_2\mathbb{C}$

As H-representation, $V \simeq \bigoplus_{n \in \mathbb{Z}} [V]_n$ h-eigenspace decomposition

$\cdots \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}$ acts on $[V]_n$ as multiplication by q^n.

$\rightsquigarrow \chi_V|_H = \sum_{n \in \mathbb{Z}} (\dim [V]_n) \cdot q^n$ integer coeff Laurent polynomial

— simply encodes multiplicities of weights.

Symmetric under $q \leftrightarrow q^{-1}$: function on $H/W = \mathbb{C}^*/\mathbb{Z}_2$

Chevalley restriction: $\mathbb{C}[G]^G \xrightarrow[\text{res}]{\sim} \mathbb{C}[H]^W$

(class functions determined by values on diagonalizable matrices, which must be W-invariant, & any such extends to G as polynomial in coefficients of char. polynomial)

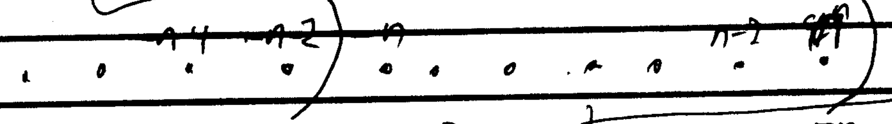
[Alternative argument: let G^{reg} = matrices w/ cyclic vector, use rational cyclic form - - - -]

⇝ characters simply encode weight multiplicities!

$$\chi_{V_n} = q^{-n} + q^{-n+2} + \dots + q^{n-2} + q^n$$

Now can encode going down in powers of q^2 via multiplication by $\frac{1}{1-q^{-2}} = 1 + q^{-2} + q^{-4} + q^{-6} + \dots$

$$\rightsquigarrow \boxed{\chi_{V_n}} = q^n\left(\frac{1}{1-q^{-2}}\right) - q^{-n-2}\left(\frac{1}{1-q^{-2}}\right)$$



$$= \frac{q^n - q^{-n-2}}{1-q^{-2}} = \boxed{\frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}}$$ Weyl character formula

We'll see a rep theoretic version of this derivation

[Note characters aren't the simplest W-invariant expressions but rather ~~of~~ the simplest skew-invariant expressions $q^{n+1} - q^{-n-1}$ over the basic skew expression $q - q^{-1} = \begin{vmatrix} 1 & 1 \\ q & q^{-1} \end{vmatrix}$ Vandermonde determinant (testing independence of diagonal values)]

Verma modules

Interesting class of ∞-dimensional $\mathfrak{g}$-modules ($=U\mathfrak{g}$-modules), constructed by Lie algebra induction

Let $\mathfrak{b} = \mathrm{Span}\{e,h\} = \mathrm{Lie}\, B \subset \mathfrak{g}$.

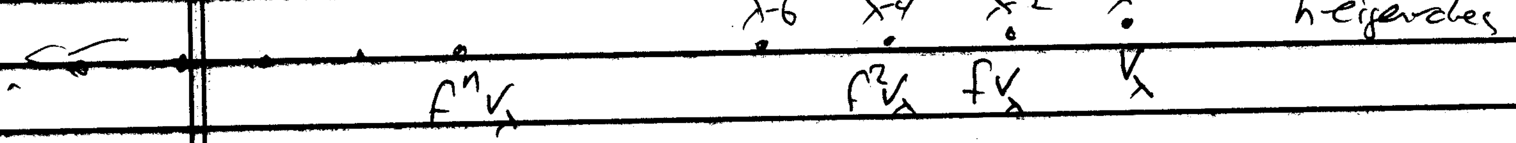
$\mathbb{C}_\lambda = \mathbb{C}\cdot v_\lambda$ 1-dim rep of $\mathfrak{b}$ with $\boxed{ev_\lambda = 0,\ hv_\lambda = \lambda v_\lambda}$

[... only integrates to B if $\lambda \in \mathbb{Z}$ ...]

$$\text{Let } M_\lambda = \mathrm{Ind}_{U\mathfrak{b}}^{U\mathfrak{g}} \mathbb{C}_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda = \frac{U\mathfrak{g} \otimes \mathbb{C}_\lambda}{xb \otimes v_\lambda = x \otimes bv_\lambda}$$

$$= U\mathfrak{g} / U\mathfrak{g}\langle e, h-\lambda\rangle$$

As a vector space $M_\lambda \simeq \mathbb{C}[f]\cdot v_\lambda$ (i.e $\simeq \mathbb{C}[f]$)
(in fact as $Un_- \simeq \mathbb{C}[f]$-module)



Note that if $\lambda = n \in \mathbb{Z}$ then M_λ is in fact an H-representation, & its formal character

$$\chi_{M_n}(q) := \sum_{k\in\mathbb{Z}} (\dim [M_n]_k)\, q^k$$

$$= q^n \cdot \frac{1}{1-q^{-2}}$$

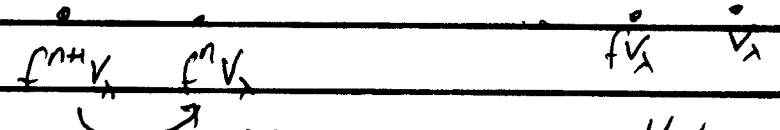
What is the structure of M_λ as $U\mathfrak{g}$-module?

We proved

$$ef^{n+1} = f^{n+1}e + (n+1)f^n(h-n)$$

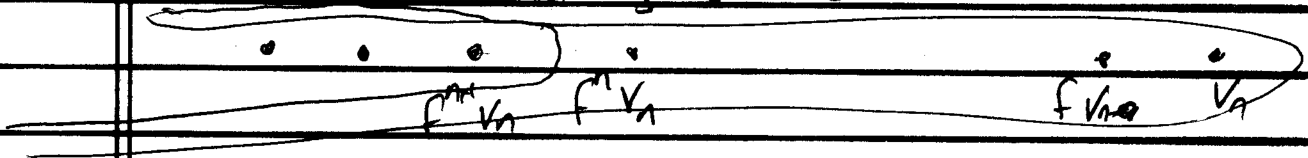
relation in $U\mathfrak{g}$

$\Rightarrow$ for $\lambda \notin \mathbb{Z}_+$ M_λ is irreducible



$(n+1)(\lambda-n)$: nonzero multiple unless $\lambda = n$

$\Rightarrow$ M_n has following structure:



$$0 \to M_{-n-2} \to M_n \longrightarrow V_n \longrightarrow 0$$ BGG resolution

short exact sequence: V_n is unique irreducible quotient of M_n & M_n indecomposable but not irreducible.

$\rightsquigarrow$ Weyl character formula $q^n \frac{1}{1-q^{-2}} - q^{-n-2}\frac{1}{1-q^{-2}}$

Harish Chandra isomorphism:

C acts on M_λ by $\frac{1}{2}((\lambda+1)^2 - 1)$ $\forall \lambda \in \mathbb{C}$

$$\rightsquigarrow Z\mathfrak{g} \longrightarrow \mathbb{C}[\mathfrak{h}^*]$$
$$A \longmapsto \{\lambda \mapsto \text{eigenvalue of } A \text{ on } M_\lambda\}$$

Image is exactly invariants of $W = \mathbb{Z}/2$ acting as reflection in -1.

Shapovalov form M_λ has a unique bilinear form $\langle , \rangle$ (/scalar) which is ~~[illegible]~~ contravariant in sense $\langle eu, v\rangle = \langle u, fv\rangle$

$\langle hu, v\rangle = \langle u, hv\rangle$

[Meaning of contragredient form: $\mathfrak{gl}_2\mathbb{C}$ has an

[Cartan] involution τ : $\tau(e) = -f$ $\tau(f) = -e$ $\tau(h) = -h$

- check relations, eg $[\tau(e), \tau(f)] = [-f, -e] = -h = \tau(h)$

(comes from $i : g \longmapsto (g^t)^{-1}$ automorphism of $Sl_2\mathbb{C}$

$$\begin{pmatrix} & e^t \\ & \end{pmatrix} \longleftrightarrow \begin{pmatrix} & \\ e^t & \end{pmatrix} \qquad \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \longleftrightarrow \begin{pmatrix} e^{-t} & \\ & e^t \end{pmatrix}$$

fixed points $(Sl_2\mathbb{C})^i = SO_2\mathbb{C}$

Contragredient form: $\langle x \cdot v, w \rangle + \langle v, \tau(x) \cdot w \rangle = 0$

twisted invariance wrt involution.]

$\langle , \rangle$ on M_λ uniquely determined by $\langle v_\lambda, v_\lambda \rangle = 1$

eg $\langle f v_\lambda, f v_\lambda \rangle = \langle v_\lambda, ef v_\lambda \rangle = \langle v_\lambda, \underset{0}{fe}\, v_\lambda \rangle + \underset{=1}{\langle v_\lambda, h v_\lambda \rangle}$

Ker $\langle , \rangle$ is precisely vectors which cannot be brought to v_λ under $Un = \mathbb{C}[e]$ action

[Lemma Ker $\langle , \rangle \subset M_\lambda$ is the maximal submodule of V_λ]

$$\rightsquigarrow \qquad 0 \to \mathrm{Ker}\, M_\lambda \to M_\lambda \longrightarrow L_\lambda \to 0$$

unique irreducible quotient

$L_\lambda = M_\lambda$ unless $n = \lambda \in \mathbb{N}$, Ker $M_n = M_{n-2}$,

$L_n = V_n$.

Contragredient Verma Module

$$M_\lambda^\vee = \text{restricted dual of } M_\lambda = \bigoplus_k [M_\lambda]_k^*$$

Sum of duals of all graded pieces = union of duals to the subspaces of bdd weights

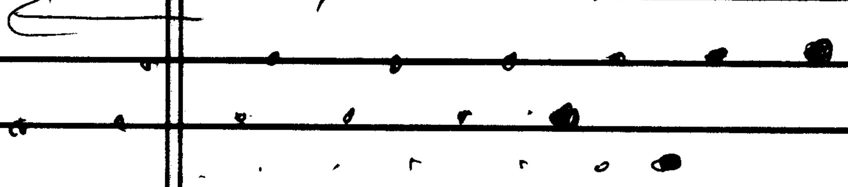
Make $M_\lambda^\vee$ into a $U\mathfrak{g}$ module as contragredient representation : $V = \bigoplus [V]_k$ $U\mathfrak{g}$ rep

$\rightsquigarrow V^\vee = \bigoplus [V]_k^*$ with action

$\langle x.v^\vee, v \rangle = - \langle v^\vee, \tau(x) \cdot v \rangle$

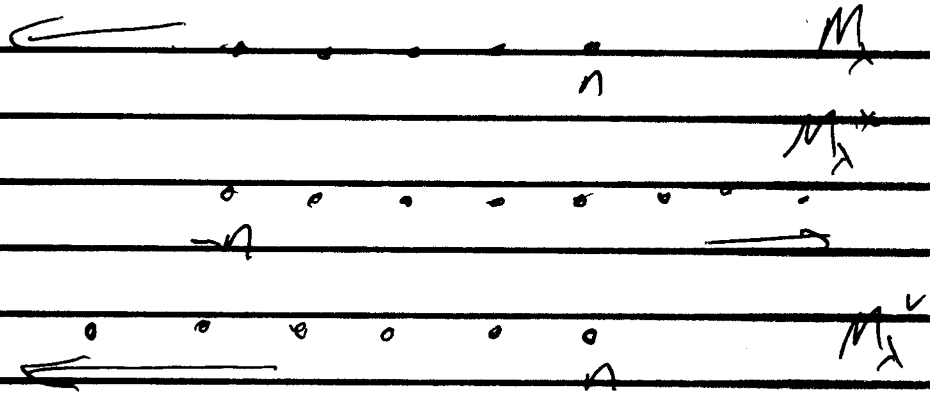
What's the point?

Def V a $U\mathfrak{g}$-module is a highest weight module (or V is an object of Category $\mathcal{O}$) if $V = \bigoplus [V]_\lambda$, ie $\mathfrak{h}$ acts semisimply, each $[V]_\lambda$ is finite dimensional & for every $v \in V$ have $e^N . v = 0$ $N \gg 0$



If $V \in \mathcal{O} \Rightarrow V^\vee \in \mathcal{O}$ as well.

$M_\lambda^\vee$ is a highest weight module



Shapovalov form defines a rep of representations

$(0 \to \mathrm{Ker}\langle,\rangle \to) M_\lambda \longrightarrow M_\lambda^\vee$, isomorphism for $\lambda \notin \mathbb{N}$

$\underline{\lambda = n}$ $\quad \mathrm{Ker}\langle,\rangle = M_{-n-2} \to \boxed{\begin{matrix} M_n & \xrightarrow{\langle,\rangle} & M_n^\vee \\ & \twoheadrightarrow V_n = V_n^\vee & \end{matrix}}$ irrep is $\underline{\text{quotient}}$ of M_n & $\underline{\text{Sub}}$ of $M_n^\vee$.

$\rightsquigarrow M_n^\vee \not\simeq M_n$.

How to understand $M_\lambda^\vee$?

M_λ has defining property: $\underline{\text{generated}}$ by h.w. vector v_λ. In fact it is free with this property:

V any Ug rep & $w \in V$ a highest weight vector, ie $h \cdot w = \lambda w$, $e \cdot w = 0$

$\Longrightarrow \exists! \; M_\lambda \longrightarrow V$

$\qquad\qquad \overset{\cup\!\!\!\shortmid}{v_\lambda} \longmapsto \overset{\cup\!\!\!\shortmid}{w}$

ie $\mathrm{Hom}_{Ug}(M_\lambda, V) = V^{h.w.\lambda}$: represents functor of h.w. λ vectors

What about $M_\lambda^\vee$? it has a h.w vector $v_\lambda^\vee$ which $\underline{\text{cogenerates}}$ $M_\lambda^\vee$:

for any $v^\vee \in M_\lambda^\vee$, $\exists\, P \in Ug$, in fact $P \in Un \simeq \mathbb{C}[e]$, s.t. $P \cdot v^\vee$ is a nonzero multiple of $v_\lambda^\vee$:

if $\langle v^\vee, w\rangle \neq 0$, write $w = (\text{polynomial in } f) \cdot v_\lambda$

$\rightsquigarrow (\text{some polynomial in } e) \cdot v^\vee$ is multiple of $v_\lambda^\vee$.

M_λ^*

$\shortparallel$

$(Un_-)^* \simeq \mathbb{C}[n_-]$

$\qquad\qquad \updownarrow t$

$\mathbb{C}[f] \simeq \mathbb{C}[n^*]$ $\Longrightarrow M_\lambda^\vee \simeq \mathbb{C}[f]$ as Un-module, e acts as $\frac{\partial}{\partial f}$,

$\shortparallel \qquad\quad \mathfrak{n}$

$M_\lambda^\vee$

$v_\lambda^\vee \longmapsto 1$ cogenerated by 1.

– special case of $(\mathbb{C}[V])^* \simeq \mathcal{D}_V = \mathcal{U}(V)$ duality...