Towards Harish Chandra Modules Why study $U\mathfrak{g}$-modules?

Idea: can capture essence of ∞-dim rep theory of a red. Lie group G algebraically in terms of 1. action of $U\mathfrak{g}_{\mathbb{C}}$: algebraic data
2. action of maximal compact subgroup $K \subset G$: discrete combinatorial data.

__Theorem__ K compact Lie group, W continuous rep of K on locally convex topological vector space (eg Hilbert, Banach, Frechet..) $\Rightarrow$

$$W^{fin} := \{w \in W : w \text{ contained in a f.d. } K\text{-invariant subspace}\} \subset W \text{ is dense}$$

Theorem in case $W = C(K)$ continuous functions ~~&~~ Peter-Weyl theorem:

$$C(K)^{fin} = \bigoplus_{V \text{ irrep}} V \otimes V^* \quad : \text{we've seen finite vects are spanned by matrix elts}$$

Peter-Weyl says this is dense in $C(K)$.

Let's deduce the theorem from Peter-Weyl:
Recall $C(K)$ has algebra structure under convolution (group algebra),

$$f * g = \int_K f(hk^{-1}) g(k)\, dk$$

We define our class of modules to be those in which group algebra $C(K)$ acts: want

$$f * w := \int_K f(k)\, k \cdot w \, dk \in W \text{ to make sense}$$

Claim $f \in C(K)^{fin}, w \in W \Rightarrow f * w \in W^{fin}$

Pf $g \in K \quad g \cdot f * w = (g \cdot f) * w$

So if $\mathrm{span}\{g \cdot f : g \in K\}$ f.d so is

$\mathrm{span}\{g \cdot (f * w) : g \in K\}$

If we had an element $\delta_1 \in C(K)$ we'd argue $W \ni w = \delta_1 * w = \lim_{n \to \infty} f_n * w \in \overline{(W^{fin})}$

for a sequence $f_n \in C(K)^{fin}$ converging to δ_1

But in any case can approximate δ_1 arbitrarily closely by bump fns $\int f = 1$ supported near $1 \Rightarrow$

$Id \in End(W)$ is in the closure of $\mathrm{Image}(C(G)) \Rightarrow$ in closure of $\mathrm{Image}(C(G)^{fin}) \Rightarrow \overline{W^{fin}} = W$.

Thus a rep of K has a combinatorial aspect: which (f.d) irreps appear, & with what multiplicity,

& an analytic aspect: which topology should we complete W^{fin} in to get W?

Two largely independent theories!

G any Lie group, $K \subset G$ maximal compact, $W \circlearrowleft G$

$\Rightarrow W^{fin} \subset W$ dense subspace of K-finite...

but G doesn't act

Def G Lie group $\curvearrowright W$ topological vector space

$w \in W$ is **smooth** if $G \longrightarrow W$ is a smooth map
$g \longmapsto g \cdot w$

- ie pullback of linear functions are smooth functions on G.
  - Collection of smooth vectors: $W^\infty \subset W$

Prop $W \longrightarrow W^\infty$ defines a functor G-reps $\longrightarrow$ Smooth G-reps

Proof $W^\infty \subset C^\infty(G, W)$. Claim it's a closed subspace

$\Rightarrow G \times W^\infty \to W^\infty$ continuous (in fact smooth)

Let $w_i \in W^\infty$, want to calculate $\lim w_i$.

Let $f_i \in C^\infty(G, W)$ be corresponding functions & suppose $f = \lim f_i$ exists. Take $w = f(1) \in W$

check $w = \lim w_i \Rightarrow \lim w_i$ is smooth ($f \in C^\infty$) ∎

Advantage of smooth representations: $\mathfrak{g}$ acts!

$w \in W$ smooth rep, $x \in \mathfrak{g} \Rightarrow$ define

$$x \cdot w = \frac{d}{dt}\Big|_{t=0} (e^{tx} \cdot w) = \lim_{t \to 0} \frac{e^{tx} w - w}{t} \in W$$

Theorem $W^\infty \subset W$ is dense.

Lemma 1 $V \subset W$ subspace closed under $x \in \mathfrak{g}$ $w \in V \longmapsto$

$$x \cdot v := \lim_{t \to 0} \frac{\pi(\exp tx) v - v}{t} \Rightarrow V \subset W^\infty$$

Pf Let $f_v : G \longrightarrow V$ $f_v(g) = g \cdot v$. By hypothesis

f_v differentiable at $1 \in G$.

$$g \in G: \quad f_v'(g) = \lim_{t \to 0} \frac{f_v(\exp(tx) g) - f_v(g)}{t} = g \cdot \lim_{t \to 0} \frac{g^{-1}\exp(tx) g \cdot v - v}{t}$$

$x^g = g^{-1} x g$
$$= g \cdot (x^g \cdot v)$$

Ref: W. Casselman, Continuous Representations
math.ubc.ca/~cass/research/pdf/Continuous.pdf

Harish Chandra modules Combine smoothness & finiteness:
Restrict to G real reductive ... eg $G = SL_2\mathbb{C}, SL_2\mathbb{R}$
& $K \subset G$ its maximal compact subgroup ... $K = SU_2$ or SO_2
.... in particular G is homotopic to K...
Want to say $C(K)^{fin} \subset C_c^\infty(G)$ & hence
K-finite vectors are automatically smooth....
Now $C(K)^{fin} \subset \mathbb{C}[G]$ algebraic/$\mathbb{C}$-analytic
but not compactly supported... however!

Prop If W is an admissible representation of G,
ie K-isotypic components are finite dimensional
[every K irrep appears finitely often]
& $w \in W^{K\text{-fin}} \Rightarrow \exists f \in C_c^\infty(G)$ st $f \cdot v = v$

Pf. Let η be an idempotent in $C(K)$ which is identity
on $U = \mathbb{C}(K)\cdot w \subset W^{fin}$ (fin dim subspace)
(eg η made out of characters of K-reps in U)
$\Rightarrow \pi(\eta)\pi(C_c^\infty(G))\pi(\eta) \subset \mathrm{End}_{\mathbb{C}} U$
(f.d.) closed subalgebra.
But we can approximate δ_1 in $C_c^\infty(G)$
$\Rightarrow$ can approximate Id_U in this subalgebra
$\Rightarrow$ Id_U is the image of some $f \in C_c^\infty(G)$. ∎

Corollary W admissible $\Rightarrow W^{fin} \subset W^\infty$
& is invariant under $\mathfrak{g}$
[In fact: $W^{fin} \subset W^\omega$ real analytic: $G \xrightarrow{\cdot w} W$ is real analytic]
[Note $U \subset W^{fin}$ f.d., action $\mathfrak{g} \otimes U \to \mathfrak{g}\cdot U \subset W^\infty$ K-equivariant
$\Rightarrow$ K-finite.

Def Suppose given • a Lie algebra $\mathfrak{g}$ • a Lie group K
• an embedding $\mathfrak{k} = \mathrm{Lie}\, K \subset \mathfrak{g}$ • an action $K \circlearrowright \mathfrak{g}$
of K on the Lie algebra extending adjoint action on $\mathfrak{k}$

A $(\mathfrak{g}, K)$ module is a vector space V with
{ • $\mathfrak{g}$ (or $U\mathfrak{g}$) module structure
• action of K

$\mathfrak{g} \circlearrowright V \circlearrowleft K$, $\mathfrak{k}$

s.t. 1. two resulting actions of $\mathfrak{k}$ agree
2. $k(x \cdot v) = (\mathrm{Ad}\, k \cdot x)\, k \cdot v \quad v \in V,\ x \in \mathfrak{g},\ k \in K$

$\Rightarrow$ W a continuous rep of G which is admissible $\longmapsto W^{fin}$ a $(\mathfrak{g}, K)$ module
$K \subset G$ max cpt.

Def A Harish Chandra module is a $(\mathfrak{g}, K)$ module st 1. K-admissible 2. $\mathfrak{g}$ fin gen.

$\boxed{\text{Lemma}}$ (Harish Chandra) 1. W irreducible unitary $\Rightarrow$
W is admissible, & 3. $W \longmapsto W^{fin}$ embeds
unitaries as a full subcategory of $(\mathfrak{g}, K)$ modules:
don't lose anything on unitaries.

2. W^{fin} consists of ~~analytic~~ vectors (for V Banach space)

Pf W admissible $\Rightarrow$ passing to K-finite vectors
gives a bijection
$\{\text{closed } G\text{-subspaces of } W\} \longleftrightarrow \{(\mathfrak{g}, K)\text{-submodule of } W^{fin}\}$

Pf of 3 $\quad W \subset V$ closed subrep $\rightsquigarrow W^{fin} \subset V^{fin}$
$\overline{W^{fin}} \subset \overline{V^{fin}}$ (with $W = \overline{W^{fin}}$, $V = \overline{V^{fin}}$)

Need to check: $M \subset V^{fin}$ $(\mathfrak{g}, K)$-submodule $\Rightarrow \overline{M} \subset V$
is G-invariant, ie $g \in G$, $\pi(g) \cdot M \subset \overline{M}$
$\Leftrightarrow \langle \lambda, g \cdot M \rangle = 0 \quad \forall \lambda \in M^{\perp} \subset V^*$. But $\langle \lambda, g \cdot m \rangle$
is an analytic function $\Rightarrow$ need only check Taylor series $\rightsquigarrow U\mathfrak{g}$ action..
but M is $U\mathfrak{g}$ invariant!! ∎

Our main example: $G = SL_2\mathbb{R}$, $K = SO_2$ max compact

A $(\mathfrak{g}, K)$ module is a representation of $\mathfrak{g}$ s.t. K acts integrably ... ie h eigenvalues are all integers!

Principal Series

$$SL_2\mathbb{R} \supset B_{\mathbb{R}} = M \quad A \quad N$$
$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \begin{pmatrix} \pm 1 & \\ & \pm 1 \end{pmatrix} \begin{pmatrix} a>0 & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$
$$\mathbb{Z}/2 \times \mathbb{R}_+ \times \mathbb{R}$$

Irreps of $M \times A (= B/N)$ labeled by $h \in \{\text{triv}, \text{sign}\}$ & $s \in \mathbb{C}$
$a \mapsto a^s = e^{s \log a}$ on $\mathbb{R}_+$

$$V_{b,s} = \mathrm{Ind}_{B_{\mathbb{R}}}^{SL_2\mathbb{R}} \mathbb{C}_{b,s} = \left\{ F \in C^\infty(SL_2\mathbb{R}) : F(gman) = F(g)(-1)^b a^s \right\}$$

$SL_2\mathbb{R}/B_{\mathbb{R}} = \mathbb{R}P^1$ $\quad$: $SL_2\mathbb{R}/N = \mathbb{R}^2 \setminus 0$,
$SL_2\mathbb{R}/AN = S^1$, $SL_2\mathbb{R}/MAN = S^1/\mathbb{Z}_2 = \mathbb{R}P^1$

$\rightsquigarrow V_{b,s}$ = smooth sections of line bundle on $\mathbb{R}P^1$
$SL_2\mathbb{R} \times_{B_{\mathbb{R}}} \mathbb{C}_{b,s}$

Recall an s-form is an expression $f(x)\,dx^s$, transforms as
$$g \cdot (f(x)\,dx^s) = f(g^{-1}x)\,(d(g^{-1}x))^s$$
$$= f(g^{-1}x)\,((g^{-1})')^s\,dx^s$$
s-densities: $g \cdot (f(x)\,|dx|^s) = f(g^{-1}x)\,|(g^{-1})'|^s\,|dx|^s$

$f(x)\,|dx|^s$

Note $\begin{matrix} k^* \to \mathbb{A}^2 - 0 \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$ is line bundle $\mathcal{O}(-1)$, cotangent bundle is $\mathcal{O}(-2)$ so $\mathcal{O}(-1) = \omega^{\frac{1}{2}}$

$k = \mathbb{R}, \mathbb{C}$

$\Rightarrow$ ~~sections~~ linear functions on $\mathcal{O}(-1)$ are $-\frac{1}{2}$-forms

$\rightsquigarrow V_{b,s}$ = sections of $\mathcal{O}(1)^{b,s}$ $\quad ((b,s) : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ homomorphism$)$

topologically: trivial | Möbius $\quad = \begin{cases} s\text{-densities} & b = \text{triv} \\ s\text{-forms} & b = \text{sign} \end{cases}$

Concretely: $\bar{N} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \subset G_{\mathbb{R}}$ "transverse" to B

$(\bar{n} \oplus b = g) \rightsquigarrow \bar{N} B_{\mathbb{R}} \subset G_{\mathbb{R}}$ open,

inverse image of open orbit $\underset{\cong \mathbb{R}}{\bar{N} \cdot \infty} \subset \mathbb{RP}^1$

$F \in V_{b,s} \rightsquigarrow$ restrict to $\mathbb{R}$ get function $f \in C^\infty(\mathbb{R})$

$$f(x) = F\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

$$G \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad g^{-1}x = g^{-1}\begin{pmatrix} 1 \\ x \end{pmatrix} = \frac{ax - c}{-bx + d}$$

$$(g^{-1})' = \frac{1}{(-bx+d)^2} \quad \text{since } ad - bc = 1.$$

Calculate $(g \cdot f)(x) = F\left(g^{-1}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) = F\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right)$

$$= F\begin{pmatrix} d - bx & -b \\ -c + ax & a \end{pmatrix} = F\left(\begin{pmatrix} 1 & 0 \\ \frac{ax-c}{-bx+d} & 1 \end{pmatrix}\begin{pmatrix} d - bx & -b \\ 0 & \frac{1}{d-bx} \end{pmatrix}\right)$$

$$= F\left(\begin{pmatrix} 1 & \\ \frac{ax-c}{-bx+d} & 1 \end{pmatrix}\right) (-1)^{b(d-bx)} |d - bx|^s$$

$$= f(g^{-1}x)\, (-1)^{b(d-bx)}\, |(g^{-1})'|^{-\frac{s}{2}} \qquad -\frac{s}{2} \text{ density or form}$$

Action of e, f, h: $f \in \mathrm{Lie}\,\overline{N}$ acts as $-\frac{d}{dx}$

$$(e^{tf}\cdot f)(x) = f(x-t)$$

$$(e^{te}\cdot f)(x) = f\left(\begin{pmatrix}1 & -t\\ & 1\end{pmatrix}\begin{pmatrix}1\\ x\end{pmatrix}\right)$$

$$= f\left(\frac{x}{-tx+1}\right)\underbrace{(\pm 1)^{b(-tx+1)}}_{=1 \text{ for } t \text{ small}}\underbrace{|-tx+1|^{s}}_{=(-tx+1)^s \ t \text{ small}}$$

$$\Rightarrow (e\cdot f)(x) = f'(x)x^2 - sx\, f(x)$$

$$\boxed{\begin{aligned} e &= x^2\frac{d}{dx} - sx\\ h &= 2x\frac{d}{dx} - s\\ f &= -\frac{d}{dx}\end{aligned}}$$

s shifted version of usual action on functions on $\mathbb{P}^1$

... $\mathfrak{sl}_2$ action on $-\frac{s}{2}$ densities via Lie derivative.

Casimir $2ef + 2fe + h^2 = 2(x^2\partial - sx)(-\partial)$
$+ 2(-\partial)(x^2\partial - sx) + (2x\partial - s)^2$
$= s^2 + 2s = \lambda^2 - 1$ for $\boxed{\lambda = s+1}$
$s^2+2s+1 = (s+1)^2$

[$\rightsquigarrow$ for $s \notin \mathbb{Z}$ $V_{s,\varepsilon}$ irreducible]

To describe h eigenspaces (& hence identify $(\mathfrak{g}, K)$ module) better to change pictures.

Iwasawa decomposition: $G = KAN = SO_2\, B_{\mathbb{R}}^{+}$ positive diagonal
(Gram-Schmidt) ... $K \cap B_{\mathbb{R}} = M$

$$\rightsquigarrow \begin{array}{ccccc} G/AN & \cong & K & \cong & S^1\\ \downarrow & & \downarrow & & \downarrow\\ G/B & \cong & K/M & \cong & \mathbb{RP}^1\end{array}$$

From this POV, note $C^\infty(K) \simeq$ fns on $\mathbb{R}^2 \setminus 0$ which are s-homogeneous any $s \in \mathbb{C}$

$\Rightarrow V_{b,s} \simeq \{ \tilde{F} : K \to \mathbb{C} :$
$$\tilde{F}(km) = (-1)^{b(m)} \tilde{F}(k) \}$$

either even or odd smooth functions on circle

ie $V_{b,s} \underset{\text{dense}}{=} \begin{cases} \bigoplus_{n \text{ even}} \mathbb{C}_n & b = \text{triv} \\ \bigoplus_{n \text{ odd}} \mathbb{C}_n & b = \text{sgn} \end{cases}$

$\mathbb{R} \cup \infty$ "distorted" picture of $K \simeq S^1$:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix} \Longrightarrow x = -\tan\theta$$
$$(\theta = -\arctan x)$$

$d\theta = \frac{-dx}{1+x^2}$ is K-invariant measure

(dx is N-invariant, not so useful!)

$e^{in \arctan x} (1+x^2)^{s/2} |dx|^{-s/2}$

$f_n = e^{in\theta} (d\theta)^{s/2}$ // K-eigenbasis

This basis not adapted to our e, f, h but rather to $SU(1,1)$ version (ie replace $\mathbb{R} \cup \infty$ by unit circle)

$H = i\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i(e - f)$

$X = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = \frac{1}{2}(h - i(e+f))$

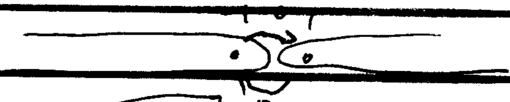
$Y = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \frac{1}{2}(h + i(e+f))$

$\Bigg\}\ \bar{A} + A^t = 0$

$H f_n = n f_n$

Lemma $X f_n = -\frac{n-s}{2} f_{n+2}$ $\quad Y f_n = \frac{n+s}{2} f_{n-2}$

⇒ $s=-1$ odd



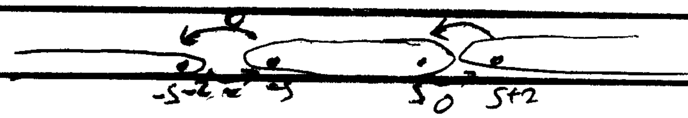
$\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$

$(-\frac{s}{2}=\frac{1}{2})$ $\boxed{\tfrac{1}{2}\text{ form}}$ to densities

sum of two irreps

$n=s>-1$:

$b=\mathrm{sgn}(s)$



$\mathbb{C} \rightarrow \mathbb{C} \leftarrow \mathbb{C}$

has fd rep as sub:

$V_{\mathrm{sgn}(n),n} = C^\infty$ sections of $\mathcal{O}(n)$

$\cup$

polynomial sections of $\mathcal{O}(n)$ =

homog polynomials of deg n on $\mathbb{R}^2$ $= V_n$ fd. $SL_2\mathbb{C}$ rep

$n=s<-1$

$b=\mathrm{sgn}(s)$

$\mathbb{C} \xleftarrow{\sim} \mathbb{C} \xrightarrow{\sim} \mathbb{C}$

has fd as a quotient

Note duality $s \longleftrightarrow -2-s$ centered at $s=-1$:

$(\longleftrightarrow \lambda=0)$

have a canonical pairing $V_{b,s} \otimes V_{b,-s-2} \longrightarrow \mathbb{C}$

$I \mapsto I^* \otimes |\omega|$ $\quad C^\infty(\mathcal{O}(s)) \otimes C^\infty(\mathcal{O}(-s-2))$

$\downarrow$

C^∞ ~~..~~ densities $\xrightarrow{\int} \mathbb{C}$.

Casselman Subrepresentation Theorem:

Every irreducible Harish-Chandra module appears as a sub of a principal series representation

— analog of highest weight theorem for f.d. reps.