# Unitary Induction

$G \supset H \curvearrowright (V, \langle\,\rangle)$ unitary representation —

how do we induce to get a unitary rep of $G$?

eg $V = \mathbb{C}$ trivial rep

$\mathrm{Ind}_H^G = C^\infty(G/H)$ .... not unitary without

choice of a measure : can't integrate products of functions

$\int$ makes sense on densities on a manifold

$f\,|dx|$ ... or top forms in oriented case

(Densities $|\Lambda^{top} T^* X|$ )

$X = G/H \qquad T_H^* G/H = (\mathfrak{g}/\mathfrak{h})^*$

$\rightsquigarrow$ densities are sections of $|\Lambda^{top}\, \mathfrak{g}/\mathfrak{h}|$ :

$G$-equivariant bundle associated to representation

$\delta = \delta_{G/H} : h \mapsto |\mathrm{Det}_{\mathfrak{g}/\mathfrak{h}}\, \mathrm{Ad}^* h|$

Modular character $H \to \mathbb{R}_+$

$= |\mathrm{Det}_{\mathfrak{h}}\, h| \;/\; |\mathrm{Det}_{\mathfrak{g}}\, h|$

$= \delta_G(h) \;/\; \delta_H(h)$

Case of $SL_2\mathbb{R}/B$ :

$\delta\begin{pmatrix} a & b \\ & c \end{pmatrix} = |a^2|$ :

action on $\begin{pmatrix} e \\ h \end{pmatrix}$ is $\begin{pmatrix} a^2 e \\ h \end{pmatrix}$ ie $\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$

on $\begin{pmatrix} e \\ h \\ f \end{pmatrix}$ : $\begin{pmatrix} a^2 & & \\ & 1 & \\ & & a^{-2} \end{pmatrix}$ $\quad (SL_2 \xrightarrow{Ad} SO_3)$

Semisimple Lie groups are unimodular $\delta_G = 1$ :

Ad action preserves nondegenerate (Killing) form $\rightsquigarrow$ has det 1.

But Borel isn't $\rightsquigarrow \delta_{G/B} = 1/\delta_B \ldots$

$\rightsquigarrow$ to define inner product need not functions but half-densities: $L^2(X) = \frac{1}{2}$-densities $|\omega|^{\frac{1}{2}}$,
can multiply & integrate

So should normalize induction to preserve unitarity:

$$\overline{\mathrm{Ind}_H^G(V, \langle , \rangle)} = \mathrm{Ind}_H^G(V \otimes \delta^{\frac{1}{2}})$$

= smooth sections of $V \otimes |\omega|^{\frac{1}{2}}$,

ie half-densities valued in $V$, completed

$$= \left\{ \begin{array}{c} F : G \to \mathbb{C} : \quad F(gh) = F(g)\,\delta^{\frac{1}{2}}(h^{-1})\,\pi_V(h^{-1}) \\ \& \int_{G/H} \langle F, F \rangle < \infty \end{array} \right\}'$$

completed in $L^2$ inner product: Hilbert space.
(inner product $\langle F, G \rangle$ is an integrable density)

So which principal series get unitary structure this way?

$V_s = \mathrm{Ind}_H^G \mathbb{C}_s$ : For $G$ to be unitary need $a^s \in U(1)$ for $a$ real $\Rightarrow$ need $s$ pure imaginary. ($a^s = e^{s \log a}$)

But tensoring with half densities $\leftrightarrow s = 1$, $b = \mathrm{triv}$
means we need to shift $s$ by 1:

Corollary $V_s$ is unitarizable if $\mathrm{Re}\, s = -1$.
(eg $s = -1 \rightsquigarrow \frac{1}{2}$ densities, $L^2(\mathbb{RP}^1)$ itself)
"Unitary principal series".

More generally $V_s^* \simeq V_{b,-2-s}$ via pairing

$$V_{b,s} \otimes V_{b,-2-s} \simeq |\omega| \xrightarrow{\int} \mathbb{C}$$

$$F \quad G \longmapsto \int F \otimes G$$

On the other hand, for $s \notin \mathbb{Z}$ the Harish Chandra modules for $V_s$ & $V_{-s-2}$ are isomorphic: recall $C+1 = (s+1)^2$ Casimir, & there is a unique indecomposable (in fact irrep) for $C \notin \mathbb{Z}$ ...

So "true" parameter is $C$ not $s$ ... (for irreps)

$\longleftrightarrow$ $C$ gives a function on parameter space of principal series (takes on scalar value on each) function is $\rho^{\frac{1}{2}}$ - shifted $\mathbb{Z}/2$ - invariant (ie $s \longrightarrow -s-2$).

Try to act the Weyl group $W = \{ \overset{1}{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}, \overset{w}{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}} \text{ mod } H \}$ on the principal series: $W \circlearrowright$ characters of $\begin{pmatrix} * & \\ & * \end{pmatrix}$ by conjugation, so on space of parameters for principal series. Can we lift this to reps?

No - doesn't have shift built in, $V_s \not\cong V_{-s}$!

Issue: $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} * & * \\ & * \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} * & \\ * & * \end{pmatrix} = B^-$

$\Rightarrow$ get isomorphism $\mathrm{Ind}_B^G \mathbb{C}_s \xrightarrow{\sim} \mathrm{Ind}_{B^-}^G \mathbb{C}_{-s}$,

$f \longmapsto \tilde{f}(g) = f(gw)$

$f\left(g\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|^s f(g) \Rightarrow$

$\tilde{f}\left(g\begin{pmatrix} a & \\ b & a^{-1} \end{pmatrix}\right) = f\left(g\begin{pmatrix} a & \\ b & a^{-1} \end{pmatrix} w\right) = f\left(gw \begin{pmatrix} a^{-1} & b' \\ 0 & a \end{pmatrix}\right)$

$= |a|^{-s} \tilde{f}(g).$

To get back in our principal series: impose $N$ invariance by averaging of $\tilde{f}$ (which is $N$-invariant)

$N \cong \mathbb{R}$
$dn = dx$

$$I(f) = \int_N \tilde{f}(gn)\,dn = \int_N f(gnw)\,dn$$

$$I(g\cdot f)(h) = \int_N f(g^{-1}hnw)\,dn = \int_N f(g^{-1}(hnw))\,dn$$
$$= (g\cdot I(f))(h)$$

intertwiner for $G$ actions (map of representations)

$$I(f)\left(g\cdot\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\right) = \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & b+n \\ & 1\end{pmatrix}w\right)dn$$

$$= \int_N f\left(g\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & n' \\ & 1\end{pmatrix}w\right)dn' \qquad \text{(absorb shift then}$$

$$= \int_N f\left(g\begin{pmatrix} 1 & a^2n' \\ & 1\end{pmatrix}w\underbrace{\begin{pmatrix} a^{-1} & \\ & a\end{pmatrix}}_{\to\, |a|^{-s}}\right)dn'$$

$n'' = a^2 n'$
$dn'' = a^2\,dn$

$$= |a|^{-s}\int_N f\left(g\begin{pmatrix} 1 & n'' \\ & 1\end{pmatrix}w\right)\frac{dn''}{|a|^2}$$

$$= |a|^{-s-2}\, I(f)(g) \quad \ldots\ldots \quad \text{ie land in } V_{-s-2} \ldots$$

Problem! need to prove convergence (holds for $|s|>1$, ?)
analytic continuation w/ poles
Whenever defined, for $s \notin \mathbb{Z}$ must be [0 or] an isom.

Strange consequence: can define hermitian inner product on
$s \notin \mathbb{Z}$ $V_s$ via $I: V_s \xrightarrow{\sim} \bar{V}_s^*$.
Theorem $-2 < s < 0$ this defines a (pre)unitary structure on $V_s$
$\rightsquigarrow$ complementary series of unitary representations.

Intertwiners as integral transform:

$K(x,y)$ "function" on $X \times Y$

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$\Rightarrow \quad f \mapsto (K * f)(y) = \int K(x,y) f(x)\, dx$

... really need $K$ to be density along fibers $X \times Y \downarrow Y$.

eg. $Z \subset X \times Y$ with density $\eta$ on $Z$

$$\begin{tikzcd} & Z \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\Rightarrow K_Z f(y) = \int_{(x,y)\in Z} f(x) \cdot \eta$

$= \pi_{Y*} \pi_X^* f$.

Radon transform: $\mathcal{U}_w = \{ \ell \neq \ell' \} \subset \mathbb{RP}^1 \times \mathbb{RP}^1$

$$\begin{tikzcd} & \mathcal{U}_w \arrow[dl] \arrow[dr] & \\ \mathbb{RP}^1 & & \mathbb{RP}^1 \end{tikzcd}$$

$\leadsto$ recover $I$ : $I(f)(\ell) = \int_{\ell' \neq \ell} f(\ell')\, d\ell'$

$$G \backslash G/B \times G/B \longleftrightarrow B \backslash G/B = \{\bullet, 1\} \ni w$$

$G$-invariant functions on $G/K \times G/H$

$\longleftrightarrow$ $G$-interwiners between functions on $G/K$ & $G/H$.

## Unitarity via HC modules

**Theorem** $V$ irreducible $(\mathfrak{g},K)$ module, $\langle,\rangle$ pos definite invariant inner product ($\mathfrak{g}_{\mathbb{R}}$ skew-Hermitian, $K_{\mathbb{R}}$ unitary) $\Rightarrow$ $V$ comes from a unitary representation

$\langle,\rangle \Leftrightarrow V \xrightarrow{\sim} \overline{V}^*$ necessarily unique

$\langle hv,w\rangle = -\langle v,hw\rangle$, same for $e,f$ $\rightsquigarrow$

$H = i(e-f)$, $Y = \frac{1}{2}(h+ie+if)$ $\quad X = \frac{1}{2}(h-ie-if)$ $\quad SU(1,1)$

$\Rightarrow \quad H = H^* \quad X^* = -Y.$

$\lambda \notin \mathbb{Z}$, even case: write basis as follows: $v_0 \in V_0$, $\quad v_{2n} = X^n v_0 \ \forall n$

(invertible so makes sense)

Write $\quad Yv_n = c_{n-2} v_{n-2}$, $\quad \|v_n\|^2 = a_n^2$

$$XYv_n - YXv_n = n\cdot v_n = (c_{n-2} - c_n)v_n$$

$$\underset{a_{n+2}^2}{\langle Xv_n, v_{n+2}\rangle} = \underset{-\overline{c_n}\,a_n^2}{-\langle v_n, Yv_{n+2}\rangle} \Rightarrow c_n \text{ real } \& < 0.$$

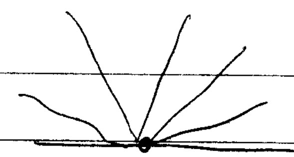
$$(C+1)v_n = \lambda^2 v_n \Rightarrow \lambda^2 v_n = v_n(1+n^2+2c_{n-2}+2c_n)$$

$$\Rightarrow \lambda^2 - n - 1 = 2(c_n + c_{n-2}) = 4c_n + 2n$$

$$\rightsquigarrow 4c_n = \lambda^2 - (n+1)^2 < 0 \quad \forall n$$

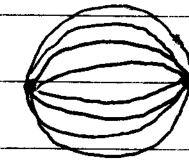
$$\Rightarrow \quad \begin{matrix}\lambda \in i\mathbb{R} \\ s \in i\mathbb{R} - 1\end{matrix} \quad \text{or} \quad \boxed{\begin{matrix}-1 < \lambda < 1 \\ -2 < s < 0\end{matrix}} \quad \text{(complementary series)}$$

Tori & "Series"   $g \in SL_2\mathbb{R}$ fall into three types:

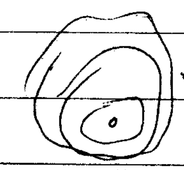


Semisimple:

- hyperbolic: $|tr\, g| > 2$
  conjugate to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ $a \in \mathbb{R}$
  two fixed points in circle

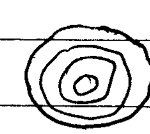


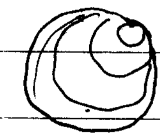
- elliptic: $|tr\, g| < 2$ conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
  fixed point in interior



unipotent:

- parabolic: $|tr\, g| = 2$ conjugate to $\pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $(\pm Id)$





single fixed point at $\infty$

Note $G \supset G^{rss}$ regular semisimple elements
($g$ s.t. $g$ is diagonalizable w/ distinct eigenvalues / $\mathbb{C}$)
has two connected components: hyperbolic & elliptic.

elliptic: diag / $\mathbb{C}$ but not $\mathbb{R}$. $Z_G(g)$ is a compact torus ($S^1$).
Over $\mathbb{C}$ centralizer is $\simeq \mathbb{C}^\times$ "complex torus".

hyperbolic: $Z_G(g)^\circ \simeq \mathbb{R}_+$ (or product for $SL_n\mathbb{R}$...)
split torus: split: diagonalizable already / $\mathbb{R}$.
Over $\mathbb{C}$ again centralizer is $\mathbb{C}^\times$:
$SL_2\mathbb{C}$ all rss elements have $\mathbb{C}^\times$ centralizer.

Harish Chandra: Representations of $G$ come in series labeled by conjugacy classes of Maximal tori. Each ~~torus~~ $T_i$ contributes series labeled by characters of $T_i$ (ie Fourier series / transforms) up to $W_i = N(T_i)/T_i$ symmetry.