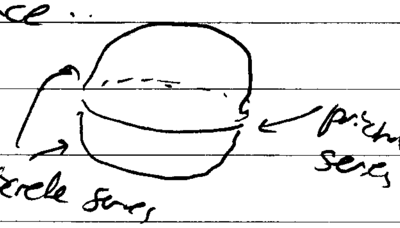
Principal series $\longleftrightarrow$ split tori (least compact in general)
Discrete series $\longleftrightarrow$ compact tori

How to build discrete series? Want to induce something from T.
... in our $SL_2\mathbb{R}$ case $T = K$ coincide..

Note $SL_2\mathbb{R} \curvearrowright \mathbb{CP}^1$ has three orbits



$\mathbb{H} \cong SL_2\mathbb{R}/K \cong$ ~~Stab~~ $\mathbb{H}$

$$\mathrm{Stab}(i) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = K$$

$n \in \mathbb{Z}$ character of $T = S^1$ $\quad \theta \mapsto e^{in\theta}$

$\rightsquigarrow$ can induce $\quad \mathrm{Ind}_T^{SL_2\mathbb{R}} \mathbb{C}_n = \left\{ \begin{matrix} F : SL_2\mathbb{R} \to \mathbb{C} \\ F(gt) = \chi_n(t) F(g) \end{matrix} \right\}$

What kind of functions? G/K is complex manifold
& these characters give holomorphic line bundles - can ask for F giving holomorphic sections:

- $SL_2\mathbb{R} \xrightarrow{2:1} PSL_2\mathbb{R} =$ unit tangent bundle of $\mathbb{H}$
  n^{th} character $\longleftrightarrow$ $n/2$ forms (sections of $\mathcal{O}(-n)$)

$$\Rightarrow \quad D_n = \{ f(z)\, dz^{n/2} : f \text{ holomorphic} \}$$

$$g \in SL_2\mathbb{R} : \underbrace{f(z)}\, dz^{n/2} \longmapsto f(g^{-1}z)\,(dg^{-1}(z))^{n/2}$$
$$= \underbrace{f\left(\frac{az - c}{-bz + d}\right)(-bz + d)^{-n}}\, dz^{n/2}$$

$\overline{D_n}$ = antiholomorphic $f \longleftrightarrow$ holomorphic on $\mathbb{H}^-$.

K-types: ~~Better~~ Write on unit disc ($SU(1,1)$ picture)

$$f(z) = \sum_{m \geq 0} a_m z^m \quad (dz^{n/2})$$

$$g = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \in K \;:\; z^m \left(dz^{n/2}\right) \longmapsto (g^{-1}z)^m \left(g^{-1\,\prime}\right)^{n/2} \left(dz^{n/2}\right)$$

$$= e^{-2mi\theta} z^m \left(e^{-2i\theta}\right)^{n/2} = e^{-(2m+n)i\theta} z^m$$

$\leadsto$ get all negative even weights starting from $-n$

$\cdots \quad -n-4 \quad -n-2 \quad -n$ ie Verma module

D_n^- take complex conjugate

Can make unitary: Poincaré measure on $\mathbb{H}$ $\quad \dfrac{dx\,dy}{y^2}$

$\mathbb{D} \quad \dfrac{|dz\,d\bar{z}|}{1-|z|^2}$

$$\Rightarrow \langle f, g \rangle = \int_{\mathbb{D}} f \bar{g} \left(1-|z|^2\right)^{n-2} |dz||d\bar{z}|$$

$$= \int_{\mathbb{H}} f \bar{g}\, y^{n-2}\, dx\, dy$$

$$= \int_{\mathbb{D}} \frac{f(z)\, dz^{n/2}\; \overline{g(z)}\, d\bar{z}^{n/2}}{\dfrac{dz^{n/2}\, d\bar{z}^{n/2}}{(1-|z|^2)^n}} \quad \frac{dz\, d\bar{z}}{1-|z|^2}$$

$\longleftarrow$ divide by invariant section to get function, which we integrate

Theorem V irrep is a closed summand of $L^2(G)$
$\quad \Longleftrightarrow V$ is a discrete series rep.

$\ldots$ discrete spectrum of Casimir on $L^2(G)$ is $\bigoplus D_n \otimes D_n^*$

Harmonic analysis on $\mathbb{H}$: $SL_2\mathbb{R} \curvearrowright L^2(\mathbb{H})$

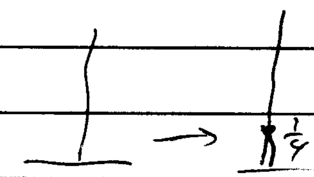
Let $\Delta = -\frac{1}{4} C = -\frac{1}{4}(H^2 + 2XY + 2YX)$

So on V_s we have $(C+1) = (s+1)^2 = s^2 + 2s + 1$

$$\Delta = -\tfrac{1}{4}(s^2 + 2s) = -\tfrac{s}{2}\left(1 + \tfrac{s}{2}\right)$$

Thus unitary principal series $s \in i\mathbb{R} - 1$

$\Rightarrow$ $\Delta \geq \frac{1}{4}$ real



Complementary series: $0 \leq \Delta \leq \frac{1}{4}$

$s = -2, 0$ $\quad s = -1$

Discrete series $s \in \mathbb{Z} \setminus \{-1\}$: Δ negative real

In coordinates $g = \begin{pmatrix} y^{\frac{1}{2}} & y^{\frac{1}{2}} x \\ & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \sin\theta \end{pmatrix}$ $\quad (g \cdot i = x + iy \in \mathbb{H})$

$$\rightsquigarrow \Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y \frac{\partial^2}{\partial x \partial \theta}$$

Thus on functions on $\mathbb{H}$ (K-invariant) get

$\Delta = \Delta_{\mathbb{H}}$ usual hyperbolic Laplacian $= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

... self-adjoint operator on $L^2(\mathbb{H})$

positive real spectrum

$\rightsquigarrow$ $L^2(\mathbb{H})_{\Delta = -\frac{s}{2}(1+\frac{s}{2})}$ is 'made up of' principal series [or complementary series for s in appropriate range of values.] --- in fact irreducible.

& spectrum is $[\frac{1}{4}, \infty)$: no complementary series!

– Poisson kernel: $f \in C(S^1) \rightsquigarrow$ extend f to harmonic function Pf on $\overline{D}$, $Pf|_{\partial D} = f$.

UHS form:

$$\cancel{Pf(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{x^2 + y^2}}$$

$Pf(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_y(x-t)\, f(t)\, dt$ $= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t) f(e^{it})\,dt$

$\frac{1+z}{1-z}$ $\quad P_y(x) = \frac{y}{x^2+y^2}$ $\quad P_r(\theta) = \mathrm{Re}\left(\frac{1+z}{1-z}\right) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}$

$0 \mapsto 1$ - uniquely characterized by being a G-intertwiner

$1 \mapsto \infty$ $\quad G/B \longrightarrow G/K \;:\; {}_B\backslash G/K \longleftrightarrow \bullet\,(/\mathbb{Z}/2)$

$-1 \mapsto 0$

$-1 \quad 0 \quad 1$ [Hardy spaces: $f \in L^p(\mathbb{R})$, Pf holomorphic]

Similar version for any eigenspace $L^2(\mathcal{H})_s$:

∞ have s-Poisson transform

$C(G/B) \longrightarrow C^\infty(\mathcal{H})_{\Delta=s}$

$Pf(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_{1-s}(x-t)\, f(t)\, dt \qquad P_{1-s} = \left(\frac{y}{x^2+y^2}\right)^{1-s}$

Let $f_s(z) = y^s = e^{s\log y}$

$\Delta f_s = s(1-s) f_s$

Any G-translate of f_s also lies in s-eigenspace

Def $\varphi_s(z) = \int_K f_s(k\cdot z)\, dk$ $\quad$ K-invariant eigenfunction

$\varphi_s(z)$ is a zonal spherical function:

matrix element of K-invariant vector in unitary rep of G,

$\varphi_s(z) = \langle v_0, g v_0 \rangle$ $\quad v_0$ spherical vector

in V_s $\quad : v_0 \in V_s^K \cong \mathbb{C}$, (even principal series)

Prop $SL_2\mathbb{R}\cdot \varphi_s(z)$ generates a copy of $V_s \subset C^\infty(G/K)$

$$\frac{e^{2t}+1}{e^{2t}-1}$$

On K invariant functions, $\Delta = -\partial_t^{\,2} - \coth t\ \partial_t$

... version of hypergeometric equation, solutions are Legendre functions

What are G-symmetries of $L^2(G/K)$?

Given by the Hecke algebra $C_c(K\backslash G/K) = C_c(G/K \times G/K)^K$

Note $K\backslash G/K \cong \mathbb{R}_+$: functions of radius,

... Cartan decomposition $G = KAK$ $\quad A \cong \mathbb{R}_+$

$K(r) \in C_c(K\backslash G/K)$, $f \in L^2(G/K) \Rightarrow$

$$K * f(z) = \cancel{\int K(|z-w|)}$$

$$\int_r \int_{|z-w|=r} K(r)\, f(w)\, dw\, dr$$

$$\begin{array}{ccc} & (G\times G/K)/K & \\ \swarrow & \downarrow & \searrow \\ K\backslash G/K & & K\backslash G/K \\ & K\backslash G/K & \end{array}$$

... smoothed out version of $\quad f \longmapsto \int_{|w-z|=r} f(w)\, dw$

average over small circle.

Gelfand trick: Hecke algebra is commutative

$\Longleftrightarrow$ (G,K) form a Gelfand pair

Pf $\quad ()^t : G \to G$ anti automorphism

$\Rightarrow$ induces anti-automorphism of $C_c(G)$

$\Rightarrow$ anti-automorphism of subalgebra $C_c(K\backslash G/K)$

But every double coset is represented by a symmetric matrix

$\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ ..... anti-automorphism is trivial

$\Rightarrow f*g = (f*g)^t = g^t * f^t = g*f$

$\Rightarrow$ $L^2(G/K)$ decomposes over the spectrum of the commutative algebra $C_c(K\backslash G/K)$ ...

Let A = closure of $C_c(K\backslash G/K)$ in $L^2(G/K)$, commutative C^* algebra $\longrightarrow$

Gelfand-Naimark $A \simeq C_0(X)$,

X = continuous $*$-homomorphisms of $A \to \mathbb{C}$ (i.e. unitary irreps of A): possible eigenvalues of A in a unitary rep. Each eigenspace will be preserved by G $\rightsquigarrow$ "direct integral" of representations.

Each $x \in X$ corresponds to $\phi_x \in L^\infty_K(G/K)$ ... in fact ϕ is a real analytic function, a spherical function associated to a spherical unitary irrep.

Harish Chandra: for $L^2(G/K)$ only need tempered representations; only see unitary principal series

$\longleftrightarrow$ $\chi: A \longrightarrow U(1)$ unitary characters $/W$

$\simeq \mathfrak{a}^*/W$

Plancherel ~~theorem~~ $L^2(K\backslash G/K) \simeq L^2(\mathfrak{a}^*/W, \mu)$

$\Gamma \subset G$ discrete subgroup: two sources

- Uniformization: any Riemann surface $g>1$ is $\simeq \Gamma\backslash\mathbb{H}$ (Fuchsian groups in compact case)
- arithmetic $\Gamma = SL_2\mathbb{Z}$ or $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0\ (N)\}$

$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 0\ (N)\}$

unit tangent bundle $= \Gamma\backslash SL_2\mathbb{R}$

$\downarrow$

$\Gamma\backslash\mathbb{H}$